OXFORD IB PREPARED

MATHEMATICS: ANALYSIS AND APPROACHES

ANSWERS

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IB DIPLOMA PROGRAMME

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IB PREPARED MAA

WORKED SOLUTIONS TO THE END-OF-CHAPTER PRACTICE QUESTIONS

Here are the worked solutions to the end-of-chapter practice questions from *IB Prepared MAA*.

For direct access, click on the name of the chapter.

1 NUMBER AND ALGEBRA: END-OF-CHAPTER PRACTICE QUESTION WORKED SOLUTIONS

SL Paper 1: Section A
No technology allowed

1. **a.** $u_1 + 2d = 4,\ u_1 + 5d = 19 \Rightarrow 3d = 15 \Rightarrow d = 5$

b. $u_1 + 10 = 4 \Rightarrow u_1 = -6$

c. $S_5 = \frac{5}{2}(2 \times -6 + 4 \times 5) = 20$

2. **a.** $u_1 r^2 = 3,\ u_1 r^5 = -24 \Rightarrow r^3 = -8 \Rightarrow r = -2$

b. $u_1(-2)^2 = 3 \Rightarrow u_1 = \frac{3}{4}$

c. $S_5 = \frac{3}{4}\frac{((-2)^5 - 1)}{(-2-1)} = \frac{33}{4}$

3. Same 2nd term $\Rightarrow 1 \times r = 0 + d \Rightarrow r = d$

$1 \times r^3 = 0 + 9d \Rightarrow r^3 = 9r$ (because $r = d$)

$r^2 = 9 \Rightarrow r = \pm 3$

$r = 3, d = 3$ or $r = -3, d = -3$

4. **a.** $\frac{u_1}{1-r} = 5u_1 \Rightarrow 1 - r = \frac{1}{5} \Rightarrow r = \frac{4}{5}$

b. $\frac{u_1}{1-r} = u_1 r \Rightarrow 1 = r(1-r) \Rightarrow r^2 - r + 1 = 0$

This quadratic has negative discriminant and thus no real solutions, showing that this is not possible.

5. **a.** $2^{4+3-6} = 2^1 = 2$

b. $3^{-9+7+1} = 3^{-1} = \frac{1}{3}$

c. $2^6 \times 1 = 64$

6. **a.**
$$\begin{aligned} \log 5^2 + \log 4 &= \log 25 + \log 4 \\ &= \log(25 \times 4) \\ &= \log 100 \\ &= \log 10^2 = 2 \end{aligned}$$

b. $\ln(7 \times 3) - \ln 21 = \ln 21 - \ln 21 = 0$

c. $e^{(\ln 7 + \ln 3) - \ln 21} = e^{\ln 21 - \ln 21} = e^0 = 1$

7. a. $\dfrac{\log 3}{\log 2}\times\dfrac{\log 4}{\log 3}\times\dfrac{\log 5}{\log 4}\times\dfrac{\log 2}{\log 5}=1$

b. $\log_2 x + 2\log_x 2 = 3 \Rightarrow \log_2 x + \dfrac{2}{\log_2 x} = 3$

Letting $\log_2 x = y$

$y + \dfrac{2}{y} = 3 \Rightarrow y^2 - 3y + 2 = 0 \Rightarrow (y-2)(y-1) = 0 \Rightarrow y = 1$ or 2

$\log_2 x = 1 \Rightarrow x = 2$, $\log_2 x = 2 \Rightarrow x = 4$

8. ${}^4C_0 1^4(-2x)^0 + {}^4C_1 1^3(-2x)^1 + {}^4C_2 1^2(-2x)^2 + {}^4C_3 1^1(-2x)^3 + {}^4C_4 1^0(-2x)^4$

$= 1 + 4(-2x) + 6(-2x)^2 + 4(-2x)^3 + 1(-2x)^4$

$= 1 - 8x + 24x^2 - 32x^3 + 16x^4$

9. a. $\text{LHS} \equiv \dfrac{(x+1)(x-1)+x(x-1)+x(x+1)}{x(x+1)(x-1)}$

$\equiv \dfrac{x^2-1+x^2-x+x^2+x}{x(x^2-1)}$

$\equiv \dfrac{3x^2-1}{x^3-x} \equiv \text{RHS}$

b. $\dfrac{3x^2-1}{x^3-x} = 0 \Rightarrow 3x^2-1=0 \Rightarrow x = \dfrac{1}{\sqrt{3}}$ or $-\dfrac{1}{\sqrt{3}}$

SL Paper 2: Section A
Technology required

10. a. $V = \dfrac{4\pi(8.5\times10^{-16})^3}{3} = 2.57\times10^{-45}\ \text{m}^3$ (3 sf)

b. $r = \sqrt[3]{\dfrac{3V}{4\pi}} = \sqrt[3]{\dfrac{3\times1.412\times10^{27}}{4\pi}} = 6.96\times10^{8}\ \text{m}$ (3 sf)

11. a. Conjecture could be an arithmetic series with $u_1 = 3$, $d = 4$

Re-check the 14 and 21 markings, expecting them to be 15 and 23.

b. Require $\dfrac{n}{2}(2\times3+(n-1)4) = 253 \Rightarrow 2n^2 + n - 253 = 0$

Solving quadratic (and rejecting negative answer) $n = 11$

12. a. Require $500(1.07)^n = 700$

Solving (e.g. graph, table, logs) gives $n = 4.97 \Rightarrow 5$ years

b. The number of vouchers forms an arithmetic series with $u_1 = 1$, $d = 1$

so $S_5 = \dfrac{5}{2}(1+5) = 15$

c. $500(1.07)^5 - 700 = £1.28$ (2 dp)

13. a. $1+{}^{20}C_1(3x)+{}^{20}C_2(3x)^2+\ldots$

$=1+20\times 3x+190\times 9x^2+\ldots$

$=1+60x+1710x^2+\ldots$

b. Term will be $2\times 60x+1\times 1710x^2=1830x^2$

14. a. General term is of the form ${}^6C_r x^{6-r}\left(\frac{2}{x}\right)^r$

This simplifies to ${}^6C_r 2^r x^{6-2r}\Rightarrow$ power of x is $6-2r$

$6-2r=4\Rightarrow r=1$

The term in x^4 is ${}^6C_1 x^5\left(\frac{2}{x}\right)=12x^4$

b. $6-2r=0\Rightarrow r=3$

The constant term is ${}^6C_3 x^3\left(\frac{2}{x}\right)^3=20\times 8=160$

HL Paper 1: Section A
No technology allowed

15. Let $P(n)$ be the statement that $2^{n+1}+3^{2n-1}$ is exactly divisible by 7.

$n=1\Rightarrow 2^2+3=7$, so $P(1)$ is true

Assume that $P(k)$ is true so that $2^{k+1}+3^{2k-1}=7s$ for $s\in\mathbb{N}$

Hence $2^{k+1}=7s-3^{2k-1}$

Now consider $P(k+1)$:

$2^{k+1+1}+3^{2(k+1)-1}=2\times 2^{k+1}+3^{2k+1}$

$=2\times\left(7s-3^{2k-1}\right)+3^{2k+1}$

$=7\times 2s+3^{2k-1}(-2+9)$

$=7\left(2s+3^{2k-1}\right)$

Since $s\in\mathbb{N}$, it follows that $2s+3^{2k-1}\in\mathbb{N}$ and hence $P(k+1)$ is true.

Since $P(1)$ is true and $P(k)$ true implies $P(k+1)$ true, by the principle of mathematical induction we have that $P(n)$ is true for all $n\in\mathbb{Z}^+$

16. a. $z=\frac{4\pm\sqrt{16-80}}{2}=\frac{4\pm\sqrt{-64}}{2}=\frac{4\pm 8\text{i}}{2}=2\pm 4\text{i}$

b. Let $w+1=z$

Then $w-3+\frac{20}{w+1}=0\Rightarrow z-4+\frac{20}{z}=0\Rightarrow z^2-4z+20=0$

So $w=1\pm 4\text{i}$

17. $\frac{2w}{1+w^2}=\frac{2\frac{z}{z^*}}{1+\frac{z^2}{(z^*)^2}}$

$=\frac{2zz^*}{(z^*)^2+z^2}$

Letting $z = a + bi$

$$\frac{2w}{1+w^2} = \frac{2(a^2+b^2)}{(a^2-2abi-b^2)(a^2+2abi-b^2)}$$
$$= \frac{2(a^2+b^2)}{(a^2+b^2)^2}$$
$$= \frac{2}{(a^2+b^2)}$$

This is purely real, giving the required result.

HL Paper 1: Section B
No technology allowed

18. a. $2x^2+5x+3=(2x+3)(x+1)$

$$\frac{2x+1}{2x^2+5x+3} \equiv \frac{A}{2x+3}+\frac{B}{x+1}$$

$\Rightarrow 2x+1 \equiv A(x+1)+B(2x+3)$

Equating coefficients:

$\Rightarrow 2 = A+2B,\ 1 = A+3B$

Solving simultaneously: $A = 4$, $B = -1$

$$\frac{2x+1}{2x^2+5x+3} \equiv \frac{4}{2x+3}+\frac{-1}{x+1}$$

b. $f(x)=4(2x+3)^{-1}-(x+1)^{-1}$

i. $f'(x)=-8(3x+3)^{-2}+(x+1)^{-2}$

ii. $\int f(x)\mathrm{d}x=\int\left(\frac{4}{2x+3}-\frac{1}{x+1}\right)\mathrm{d}x=2\ln|2x+3|-\ln|x+1|+\ c$

19. $\frac{p(2-\mathrm{i})+q(1+\mathrm{i})}{(1+\mathrm{i})(2-\mathrm{i})}=\frac{2+3\mathrm{i}}{\mathrm{i}}\times\frac{-\mathrm{i}}{-\mathrm{i}}\Rightarrow\frac{(2p+q)+(q-p)\mathrm{i}}{3+\mathrm{i}}=3-2\mathrm{i}$

$(2p+q)+(q-p)\mathrm{i}=(3+\mathrm{i})(3-2\mathrm{i})=11-3\mathrm{i}$

Equating real and imaginary parts: $2p+q=11,\ q-p=-3$

Solving simultaneously gives: $p=\frac{14}{3},\ q=\frac{5}{3}$

20. a. i. $(z+w)^*=\left((a+c)+(b+d)\mathrm{i}\right)^*=(a+c)-(b+d)\mathrm{i}=(a-b\mathrm{i})+(c-d\mathrm{i})=z^*+w^*$

ii. $(z-w)^*=\left((a-c)+(b-d)\mathrm{i}\right)^*=(a-c)-(b-d)\mathrm{i}=(a-b\mathrm{i})-(c-d\mathrm{i})=z^*-w^*$

iii. $(zw)^*=\left((ac-bd)+(bc+ad)\mathrm{i}\right)^*=(ac-bd)-(bc+ad)\mathrm{i}=(a-b\mathrm{i})(c-d\mathrm{i})=z^*w^*$

b. i. $\left(\frac{z}{w}\right)^*=\left(\frac{r}{s}\mathrm{cis}(\theta-\phi)\right)^*=\frac{r}{s}\mathrm{cis}(-\theta+\phi)=\frac{r\,\mathrm{cis}(-\theta)}{s\,\mathrm{cis}(-\phi)}=\frac{z^*}{w^*}$

ii. $\left(z^n\right)^*=\left((r\,\mathrm{cis}\,\theta)^n\right)^*=\left(r^n\,\mathrm{cis}\,n\theta\right)^*=r^n\,\mathrm{cis}(-n\theta)=\left(r\,\mathrm{cis}(-\theta)\right)^n=\left(z^*\right)^n$

21. Let $z = r\,\text{cis}\,\theta$

$\left|z^4\right| = \sqrt{2^2+2^2} = \sqrt{8} = 2^{\frac{3}{2}}$; $\left|z^4\right|$ is in the 2nd quadrant so its arg is $\dfrac{3\pi}{4}$

$(r\,\text{cis}\,\theta)^4 = r^4\,\text{cis}\,4\theta = 2^{\frac{3}{2}}\,\text{cis}\,\dfrac{3\pi}{4}$

Equating moduli and arguments:

$r^4 = 2^{\frac{3}{2}} \Rightarrow r = 2^{\frac{3}{8}}$

$4\theta = \dfrac{3\pi}{4} + 2n\pi,\ n\in\mathbb{Z} \Rightarrow \theta = \dfrac{3\pi}{16} + \dfrac{n\pi}{2}$

Giving the four roots as

$z = 2^{\frac{3}{8}}\,\text{cis}\,\dfrac{3\pi}{16},\ 2^{\frac{3}{8}}\,\text{cis}\,\dfrac{11\pi}{16},\ 2^{\frac{3}{8}}\,\text{cis}\,\dfrac{-5\pi}{16},\ 2^{\frac{3}{8}}\,\text{cis}\,\dfrac{-13\pi}{16}$

22. a. Proof by <u>contradiction</u>. Suppose $\mathbb{C}$ can be ordered.

Then either $\mathrm{i} = 0$ or $\mathrm{i} < 0$ or $\underline{\mathrm{i} > 0}$.

Case **i.** $\mathrm{i} = 0 \Rightarrow \mathrm{i}^2 = 0 \Rightarrow \underline{-1 = 0}$ (a contradiction)

Case **ii.** $\mathrm{i} < 0$ then using a property of ordering $\mathrm{i}\times\mathrm{i} > \underline{\mathrm{i}\times 0} \Rightarrow \underline{-1 > 0}$ (a contradiction)

Case **iii.** $\underline{\mathrm{i} > 0}$ then using a property of ordering $\underline{\mathrm{i}\times\mathrm{i} > \mathrm{i}\times 0} \Rightarrow \underline{-1 > 0}$ (a contradiction)

All three cases lead to a <u>contradiction</u> and so we conclude that <u>$\mathbb{C}$ cannot be ordered</u>.

b. i. no sense **ii.** sense, $3 > 5$ false **iii.** sense, $5 > 13$ false

iv. sense, $\arctan\dfrac{4}{3} > \arctan\dfrac{12}{5} \Rightarrow \dfrac{4}{3} > \dfrac{12}{5}$ false

v. no sense

23. a. $\left(\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 2 & 1 & 2 & 11 \\ 7 & -4 & 13 & \lambda \end{array}\right)$

row 2 − 2 × row1
row 3 − 7 × row1 $\left(\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 5 & -4 & -7 \\ 0 & 10 & -8 & \lambda-63 \end{array}\right)$

row 2 − 2 × row1 $\left(\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 5 & -4 & -7 \\ 0 & 0 & 0 & \lambda-49 \end{array}\right)$

So, to be consistent $\lambda = 49$

b. Let $z = t$ $\quad 5y - 4t = -7 \Rightarrow y = \dfrac{-7}{5} + \dfrac{4}{5}t$

$x - 2y + 3t = 9 \Rightarrow x = 2\left(\dfrac{-7}{5} + \dfrac{4}{5}t\right) - 3t + 9 \Rightarrow x = \dfrac{31}{5} + \dfrac{-7}{5}t$

$$\frac{x - \frac{31}{5}}{\frac{-7}{5}} = \frac{y + \frac{7}{5}}{\frac{4}{5}} = z$$

HL Paper 2: Section A
Technology required

24. a. ${}^{8}C_4 = 70$

b. ${}^{3}C_2 \times {}^{5}C_2 = 30$

25. a. There are 4 vowels so 4 possibilities for the first letter.

This leaves 6 letters which can be arranged in 6! ways,

so the answer is 4 × 6! = 2880

b. The 4 vowels can be arranged in 4! ways.

The 3 constants can be arranged in 3! ways.

It can either be the vowels first or the constants first (2 arrangements)

so the answer is 2 × 4! × 3! = 288

26. a. $$(1+2x)^{-\frac{1}{2}} = 1 + \frac{-1}{2}(2x) + \frac{\frac{-1}{2}\times\frac{-3}{2}}{1\times 2}(2x)^2 + \frac{\frac{-1}{2}\times\frac{-3}{2}\times\frac{-5}{2}}{1\times 2\times 3}(2x)^3 + \ldots$$

$$= 1 - x + \frac{3}{2}x^2 - \frac{5}{2}x^3 + \ldots$$

b. Putting $x = \frac{1}{2}$ $\quad \frac{1}{\sqrt{2}} \simeq 1 - \frac{1}{2} + \frac{3}{8} - \frac{5}{16} = \frac{9}{16}$

c. $$\frac{\left|\frac{9}{16} - \frac{1}{\sqrt{2}}\right|}{\frac{1}{\sqrt{2}}} \times 100\% = 20.5\% \text{ (3 sf)}$$

27. a. $${}^{p}C_i = \frac{p!}{i!(p-i)!} = p \times \frac{(p-1)!}{i!(p-i)!}$$

All of the prime factors of both $i!$ and $(p-i)!$ must be less than p since $1 \le i \le p-1$.

Hence, none of these prime factors can divide p.

${}^{p}C_i$ is an integer so $\frac{(p-1)!}{i!(p-i)!}$ is also an integer showing that p exactly divides ${}^{p}C_i$

b. Using part (a), the first row in Pascal's triangle that is worth considering is $n = 9$

${}^{9}C_3 = 84$ which is not exactly divisible by 9.

28. a. $x = 2, y = 3, z = 5$

b. No solutions

c. $x = 27 - 5t, y = -12 + 3t, z = t \quad \frac{x-27}{-5} = \frac{y+12}{3} = z$

HL Paper 3: Technology required

29. a. i. $x+y=3.551$ which to 3 significant figures is 3.55.

ii. $x=1.235 \to 1.24$ and $y=2.316 \to 2.32$ so adding gives 3.56

iii. The two answers are different. Adding and then rounding is not the same as rounding and then adding.

iv. Method (i) is better, as it is closer to the true answer.

b. i. $\varepsilon_{a+b}=(A+B)-(a+b)=(A-a)+(B-b)=\varepsilon_a+\varepsilon_b$

ii. $\varepsilon_{a-b}=(A-B)-(a-b)=(A-a)-(B-b)=\varepsilon_a-\varepsilon_b$

iii. The absolute maximum error is $|\varepsilon_a|+|\varepsilon_b|$ in both cases.

c. i. $\varepsilon_{a\times b}=AB-ab$

Now $AB=(a+\varepsilon_a)(b+\varepsilon_b)=ab+a\varepsilon_b+b\varepsilon_a+\varepsilon_a\varepsilon_b$ and $\varepsilon_a\varepsilon_b$ is small

so $\varepsilon_{ab}\simeq(ab+a\varepsilon_b+b\varepsilon_a)-ab=a\varepsilon_b+b\varepsilon_a$

ii. This looks similar to the product rule.

iii. The absolute maximum error is $|a||\varepsilon_b|+|b||\varepsilon_a|$

d. i. $\varepsilon_{\frac{a}{b}}=\frac{A}{B}-\frac{a}{b}=\frac{a+\varepsilon_a}{b+\varepsilon_b}-\frac{a}{b}=\frac{ab+b\varepsilon_a-ab-a\varepsilon_b}{b(b+\varepsilon_b)}=\frac{b\varepsilon_a-a\varepsilon_b}{b(b+\varepsilon_b)}$ so $\varepsilon_{\frac{a}{b}}\simeq\frac{b\varepsilon_a-a\varepsilon_b}{b^2}$

ii. This looks similar to the quotient rule.

iii. The absolute maximum error is $\frac{|a||\varepsilon_b|+|b||\varepsilon_a|}{b^2}$

e. i. $r_{a\times b}=\frac{|\varepsilon_{ab}|}{|ab|}=\frac{|a||\varepsilon_b|+|b||\varepsilon_a|}{|a||b|}=\frac{|\varepsilon_a|}{|a|}+\frac{|\varepsilon_b|}{|b|}=r_a+r_b$

ii. $r_{\frac{a}{b}}=\frac{\left|\varepsilon_{\frac{a}{b}}\right|}{\left|\frac{a}{b}\right|}=\frac{|a||\varepsilon_b|+|b||\varepsilon_a|}{|b|^2}\times\frac{|b|}{|a|}=\frac{|\varepsilon_a|}{|a|}+\frac{|\varepsilon_b|}{|b|}=r_a+r_b$

They are added in both cases.

f. i. $A^n=(a+\varepsilon_a)^n=a^n+na^{n-1}\varepsilon_a+$ (terms that can be ignored)

So, the error ε_{a^n} is $\varepsilon_{a^n}=A^n-a^n=(a^n+na^{n-1}\varepsilon_a)-a^n=na^{n-1}\varepsilon_a$

ii. The absolute relative error is $r_{a^n}=\frac{|na^{n-1}\varepsilon_a|}{|a^n|}=n\frac{|\varepsilon_a|}{|a|}=nr_a$

(which agrees with earlier work since a^n can be considered as repeated multiplication)

2 FUNCTIONS: END-OF-CHAPTER PRACTICE QUESTION WORKED SOLUTIONS

SL Paper 1: Section A
No technology allowed

1. **a.** $f'(x) = \dfrac{a(x+c)-(ax+b)}{(x+c)^2} = \dfrac{ac-b}{(x+c)^2}$

b. **i.** $ac - b > 0$

ii. $ac - b < 0$

iii. $ac - b = 0$

c. $b = ac \Rightarrow f(x) = \dfrac{ax+ac}{x+c} = \dfrac{a(x+c)}{x+c} = a, x \neq -c$

2. **a.** $y = \dfrac{x+2}{x-1}$ inverse given by $x = \dfrac{y+2}{y-1}$

$xy - x = y + 2 \Rightarrow y(x-1) = x+2$

$y = \dfrac{x+2}{x-1}$ so $f^{-1}(x) = \dfrac{x+2}{x-1}$

b. **i.** Domain $\{x \in \mathbb{R} | x \neq 1\}$ Range $\{y \in \mathbb{R} | y \neq 1\}$

ii. Domain $\{x \in \mathbb{R} | x \neq 1\}$ Range $\{y \in \mathbb{R} | y \neq 1\}$

c. $(f \circ f)(x) = x, x \neq 1$ since f is self-inverse

3. **a.** $x + c = 0 \Rightarrow x = -c$ so $c = -3$

As $x \to \pm\infty, y \to a$ so $a = 4$

$4x + b = 0 \Rightarrow x = \dfrac{-b}{4}$ so $\dfrac{-b}{4} = 2 \Rightarrow b = -8$

b. $x = 0 \Rightarrow y = \dfrac{-8}{-3}$ intercept on y-axis is $\left(0, \dfrac{8}{3}\right)$

4. **a.** x-axis being a tangent implies the function has one repeated root so

$b^2 - 4ac = 49 - 8c = 0 \Rightarrow c = \dfrac{49}{8}$

b. $2x^2 + 7x + \dfrac{49}{8} = 0 \Rightarrow 2\left(x^2 + \dfrac{7}{2}x + \dfrac{49}{16}\right) = 0 \Rightarrow \left(x + \dfrac{7}{4}\right)^2 = 0$

$x = -\dfrac{7}{4}$

5. Let $u = \sqrt{x}$ $u^2 - 7u + 10 = 0 \Rightarrow (u-2)(u-5) = 0$

$u = 2$ or 5

$x = u^2 \Rightarrow x = 4$ or 25

6. Discriminant of $(x^2 + 6x + 4) = 0$ is $36 - 16$ which is positive and so the equation has 2 real roots.

Discriminant of $(x^2 + 6x + 10) = 0$ is $36 - 40$ which is negative and so the equation has no real roots.

Hence, two real roots in total.

7. **a.** Discriminant $= 64 - 16a = 0 \Rightarrow a = 4$, the root is $\frac{-b}{2a} = \frac{-8}{2\times 4} = -1$

b. The concave up quadratic $4x^2 + 8x + (b - 10)$ cannot have any zeros so $64 - 16(b - 10) < 0 \Rightarrow b > 14$

(or by considering a vertical translation of the quadratic in part **a**.)

8. **a.** Considering the line of symmetry, $h = -2$ and then $k = -16$

$x = 0 \Rightarrow a(-2)^2 - 16 = 4 \Rightarrow a = 5$

b. $5(x-2)^2 - 16 = 0 \Rightarrow (x-2)^2 = \frac{16}{5} \Rightarrow x - 2 = \frac{\pm 4}{\sqrt{5}}$

$x = 2 \pm \frac{4\sqrt{5}}{5}$

SL Paper 1: Section B
No technology allowed

9. **a.** Gradient is $\frac{5-3}{5-1} = \frac{1}{2}$

The line $y = \frac{1}{2}x + c$ through $(1, 3)$ gives $3 = \frac{1}{2} + c \Rightarrow c = \frac{5}{2} \Rightarrow y = \frac{1}{2}x + \frac{5}{2}$

b. Gradient is -2 The line is $y = -2x + d$ through $(5, 5)$ gives $5 = -10 + d \Rightarrow d = 15$

$\Rightarrow y = -2x + 15$

c. $(2, p)$ lies on this line, giving $p = -4 + 15 = 11$

d. The line $y = \frac{1}{2}x + f$ through $(2, 11)$ gives $11 = 1 + f \Rightarrow f = 10$

$y = \frac{1}{2}x + 10$

e. The line $y = -2x + g$ through $(1, 3)$ gives $3 = -2 + g \Rightarrow g = 5$

$y = -2x + 5$

f. Solving $y = \frac{1}{2}x + 10$ and $y = -2x + 5$ $2.5x = -5 \Rightarrow x = -2, y = 9$

$D = (-2, 9)$

g. The mid-point is $\left(\frac{-2+5}{2}, \frac{9+5}{2}\right) = \left(\frac{3}{2}, 7\right)$

10. **a.** Maximum domain $\{x \in \mathbb{R} \mid x \neq 0\}$, range $\{y \in \mathbb{R} \mid y \neq 0\}$

b. Maximum domain $\{x \in \mathbb{R} \mid x \geq -5\}$, range $\{y \in \mathbb{R} \mid y \geq 0\}$

c. Maximum domain $\{x \in \mathbb{R} \mid x \neq -3\}$, range $\{y \in \mathbb{R} \mid y \neq 0\}$

d. Maximum domain $\{x \in \mathbb{R} \mid x > 3\}$, range $\{y \in \mathbb{R} \mid y > 0\}$

e. Maximum domain $\{x \in \mathbb{R} \mid x > 2\}$, range $\mathbb{R}$

f. Maximum domain $\mathbb{R}$, range $\{y \in \mathbb{R} \mid y > 0\}$

11. **a.** $y = \dfrac{x-3}{4}$ inverse given by $x = \dfrac{y-3}{4} \Rightarrow y = 4x+3$

$f^{-1}(x) = 4x+3$

b. $(f \circ f)(x) = f\left(\dfrac{x-3}{4}\right) = \dfrac{\left(\dfrac{x-3}{4} - 3\right)}{4} = \dfrac{x-15}{16}$

c. $y = \dfrac{x-15}{16}$ inverse given by $x = \dfrac{y-15}{16} \Rightarrow y = 16x+15$

$(f \circ f)^{-1}(x) = 16x+15$

d. Given any functions f and g, you know that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

So, it follows that $(f^{-1} \circ f^{-1})(x) = (f \circ f)^{-1}(x) = 16x+15$

12. **a.** $(h \circ g \circ f)(x) = h(g(f(x))) = h(g(x+4)) = h\left(\dfrac{1}{x+4}\right) = \dfrac{1}{x+4} + 3$

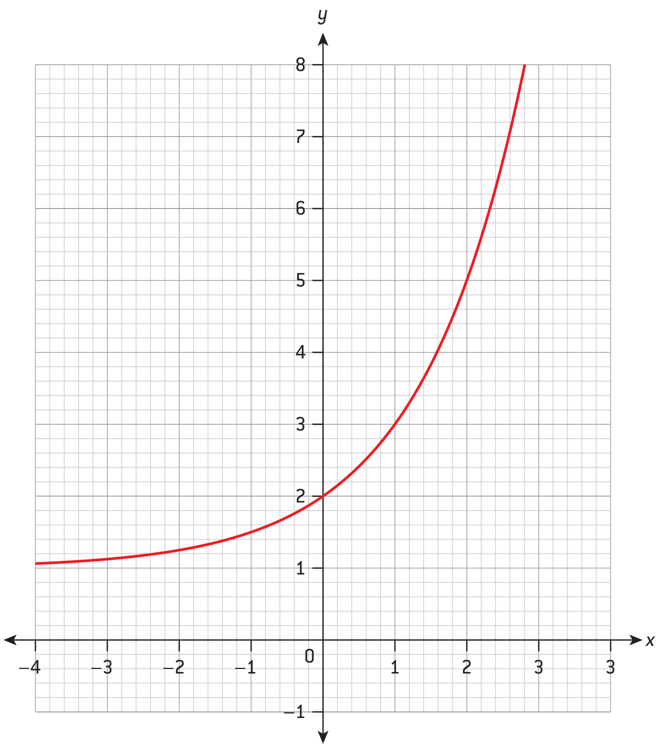
b. Domain $\{x \in \mathbb{R} | x \neq -4\}$. Range $\{y \in \mathbb{R} | y \neq 3\}$

c. $(h \circ g \circ f)^{-1}(x) = f^{-1}(g^{-1}(h^{-1}(x))) = f^{-1}(g^{-1}(x-3)) = f^{-1}\left(\dfrac{1}{x-3}\right) = \dfrac{1}{x-3} - 4$

Domain $\{x \in \mathbb{R} | x \neq 3\}$. Range $\{y \in \mathbb{R} | y \neq -4\}$

d. $\dfrac{1}{x+4} + 3 = \dfrac{3x+13}{x+4}$ $a = 3, b = 13, c = 4$

13. **a.**



Horizontal asymptote $y = 1$

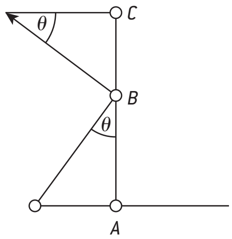
b. **i.** Vertical translation up by + 2

ii. Horizontal translation to the right by 1

iii. Horizontal stretch by a factor of $\frac{1}{4}$

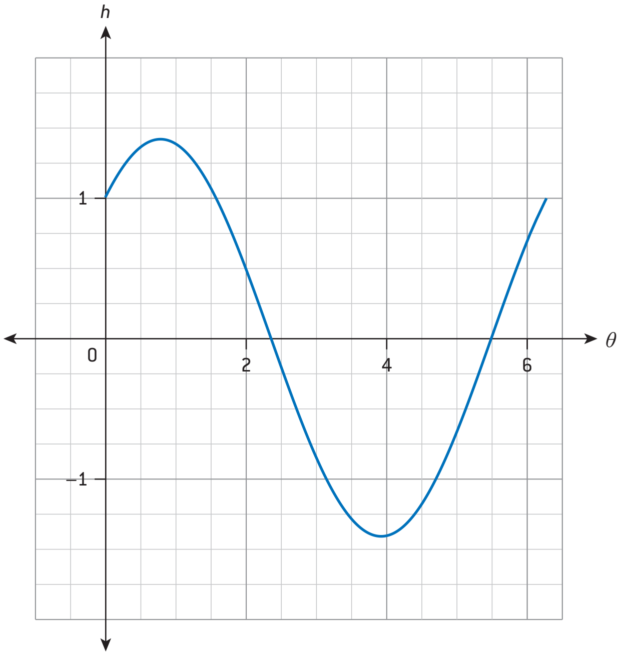
iv. $2^{x+2}+4=2^x2^2+4=4(2^x+1)$ so vertical stretch by a factor of 4

14. **a.**

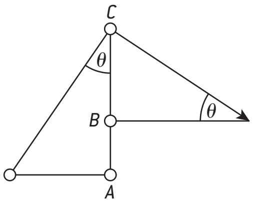


From the diagram, $h(\theta)=AB+BC=1\cos\theta+1\sin\theta=\cos\theta+\sin\theta$

b.

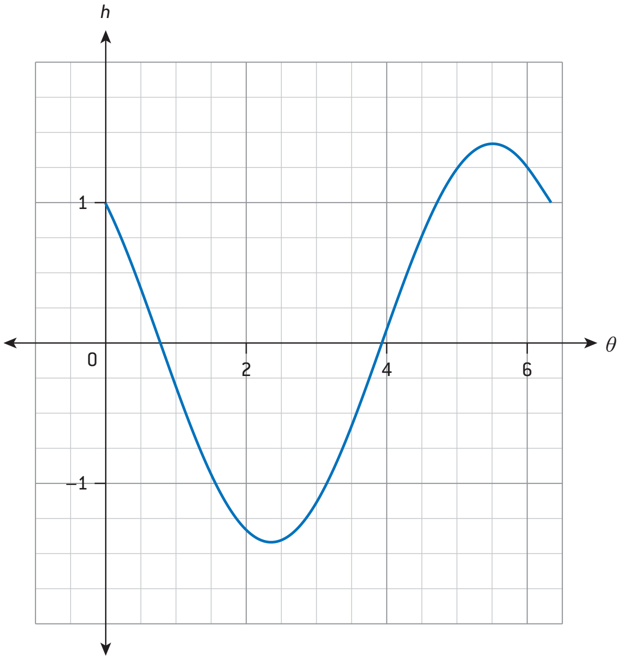


c.



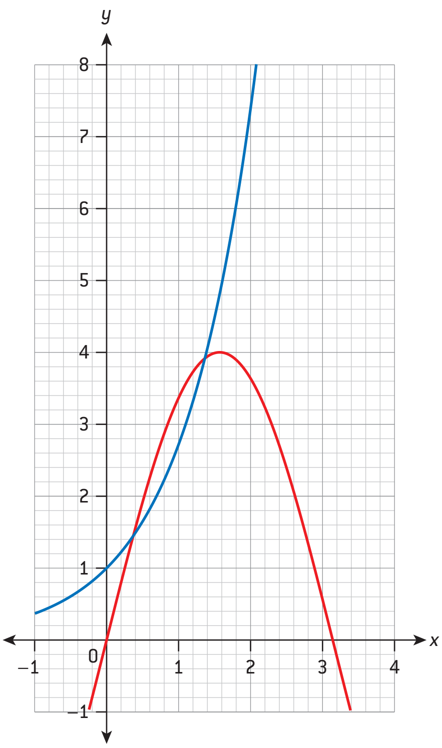
From the diagram, $h(\theta)=AC-BC=1\cos\theta-1\sin\theta=\cos\theta-\sin\theta$

d.



SL Paper 2: Section A
Technology required

15. Using a calculator to graph the functions $y = e^x$ and $y = 4\sin x$



From calculator, they intersect at $x = 0.371$ and $x = 1.37$ (3 sf)

SL Paper 2: Section B
Technology required

16. a. $g(0) = 10$, $r(0) = 1000$

b. 10 years = 1 decade $\Rightarrow g(1) = 10$

c. Solving $g(t) = 20$, $t = 1.44$, 14.4 (3 sf) years

d. Solving $g(t) = r(t)$, $t = 2.98$, 29.8 (3 sf) years

e. Solving $g(t) = 2r(t)$, $t = 3.29$, 32.9 (3 sf) years

f. Solving $r(t) < 0.5$, $t = 10.97$, 110 (3 sf) years

HL Paper 1: Section A
No technology allowed

17. a. Remainder $= p(1) = 1 + 2 - 5 - 6 = -8$

b. $q(+1) = q(-1) \Rightarrow 1 + 1 + c + 1 = -1 + 1 - c + 1 \Rightarrow c = -1$

18. a. $x^2 + 2x + 5 = (x+1)^2 + 4$, so the vertex is a minimum at $(-1, 4)$

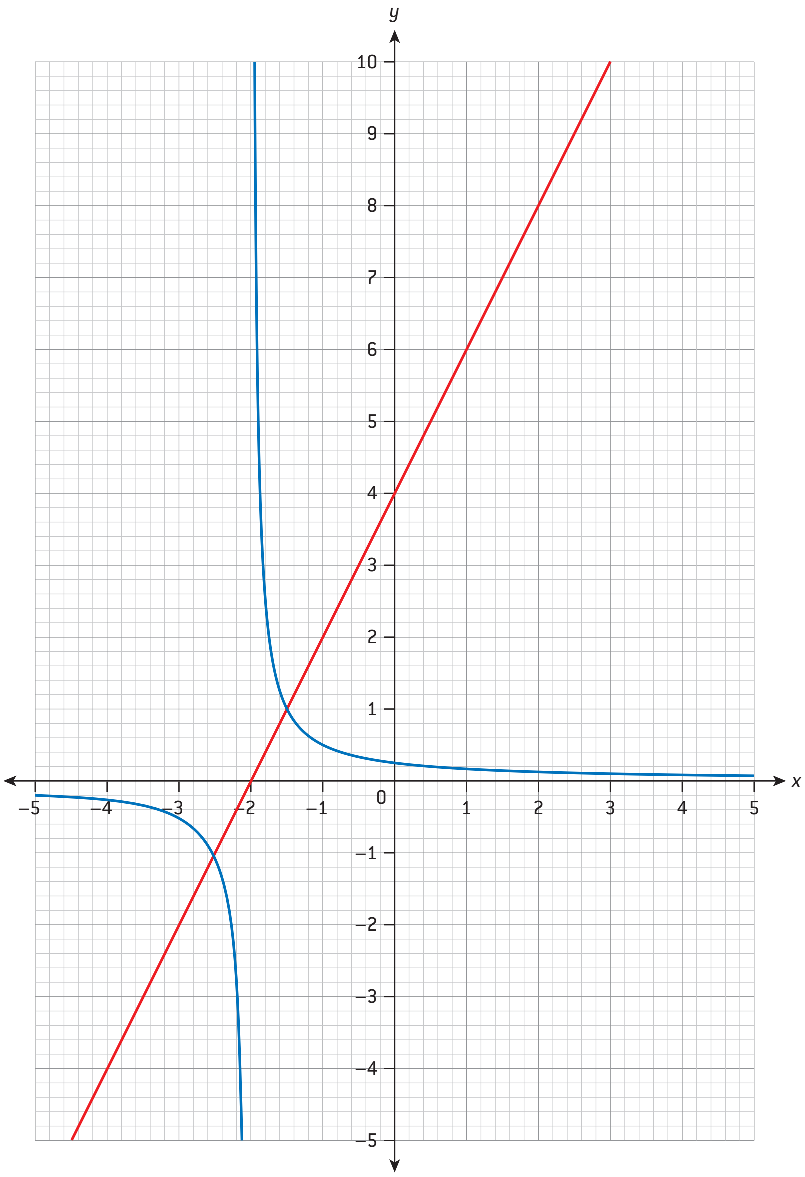
b. Maximum possible domain is $\{x \in \mathbb{R} | x \geq -1\}$

c. $f(x)$ is given by $y = (x+1)^2 + 4$

The inverse is given by $x = (y+1)^2 + 4 \Rightarrow x - 4 = (y+1)^2 \Rightarrow y = -1 + \sqrt{x-4}$

So $f^{-1}(x) = -1 + \sqrt{x-4}$; the domain is $\{x \in \mathbb{R} | x \geq 4\}$, the range is $\{y \in \mathbb{R} | y \geq -1\}$

19. **a.**



b. Intersection when $2x+4=\pm 1 \Rightarrow x=\dfrac{-5}{2}$ or $\dfrac{-3}{2}$

$f(x) > g(x)$ in the intervals $\left]\dfrac{-5}{2}, -2\right[$ and $\left]\dfrac{-3}{2}, \infty\right[$

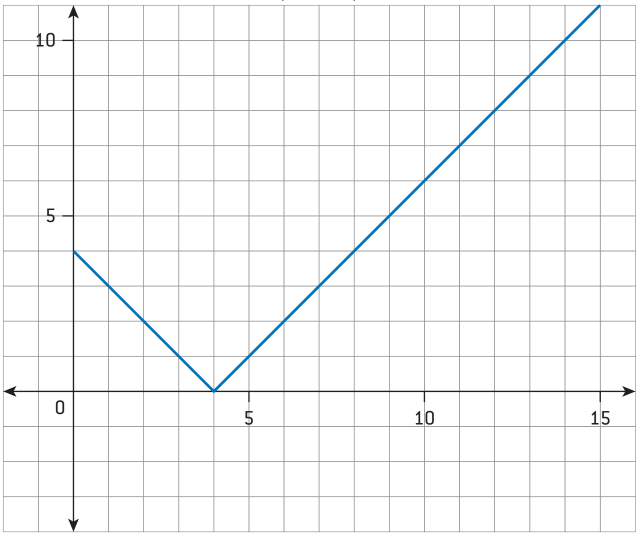
20. **a.** $y=0 \Rightarrow |x|-4=0 \Rightarrow |x|=4 \Rightarrow x=\pm 4$

So, x-axis intercepts are $(-4, 0)$ and $(4, 0)$

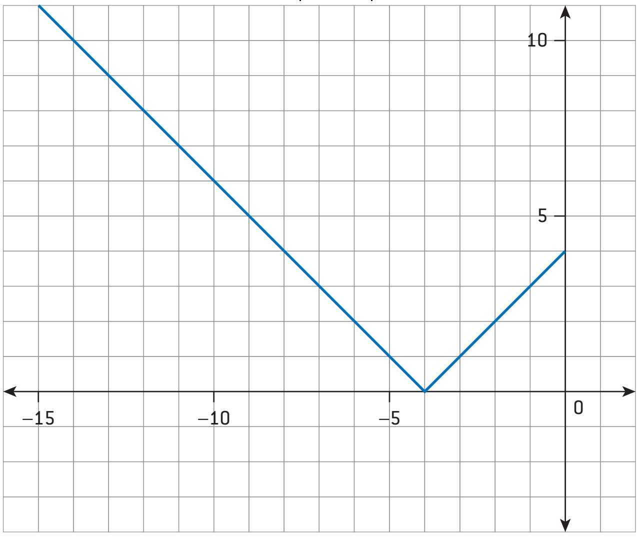
$x=0 \Rightarrow y=4$

So, y-axis intercept is $(0, 4)$

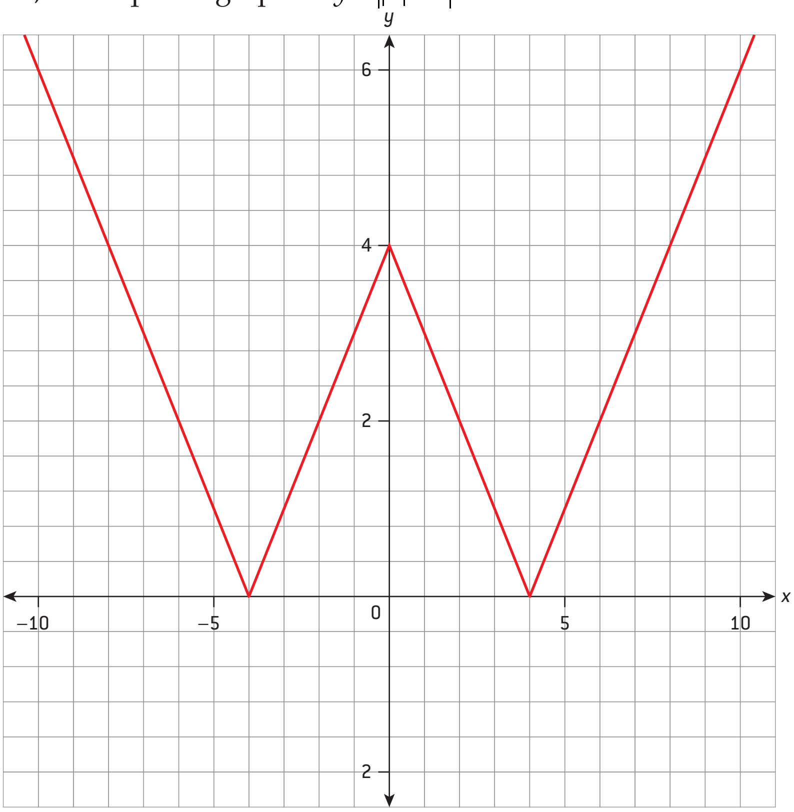
b. For $x > 0, |x| = x \Rightarrow y = ||x| - 4| = |x - 4|$



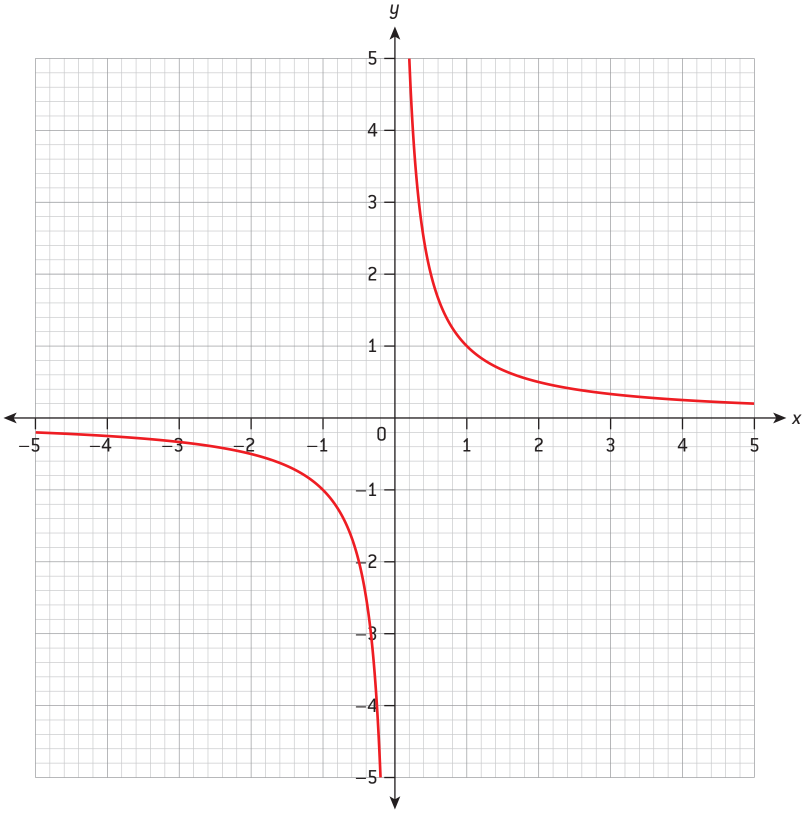
For $x < 0, |x| = -x \Rightarrow y = ||x| - 4| = |-x - 4| = |x + 4|$



So, the required graph of $y = ||x| - 4|$ will be:



21. a.



Horizontal asymptote $y = 0$; vertical asymptote $x = 0$

b. **i.** horizontal translation to the left by + 1

ii. vertical stretch by a factor of 2 (or horizontal stretch by a factor of 2)

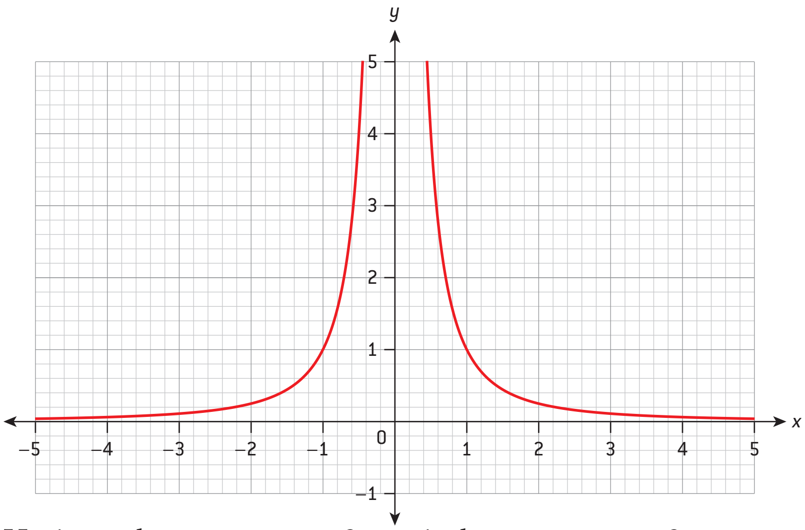
iii. horizontal stretch by a factor of $\frac{1}{3}$ (or vertical stretch by a factor of $\frac{1}{3}$)

iv. vertical translation up by +2

c. **i.** no **ii.** yes **iii.** yes **iv.** no

d. **i.** neither **ii.** odd **iii.** odd **iv.** neither

e.



Horizontal asymptote $y = 0$; vertical asymptote $x = 0$

f. **i.** not self-inverse (the inverse does not exist without a domain restriction)

ii. even function

22. **a.** **i.** $f(x) = f(-x)$

$\frac{ax+b}{x+c} \equiv \frac{-ax+b}{-x+c} \Rightarrow -ax^2 + (ca-b)x + bc \equiv -ax^2 - (ca-b)x + bc$

Equating coefficients: $ca - b = 0$

ii. $f(x) = \frac{ax+ac}{x+c}$ $\quad f(x) = a, x \neq -c$ a horizontal straight line

b. **i.** $f(x) = -f(-x)$

$\frac{ax+b}{x+c} \equiv \frac{-(-ax+b)}{-x+c} \Rightarrow -ax^2 + (ca-b)x + bc \equiv ax^2 - (b-ac)x - bc$

Equating coefficients:

$x^2 : a = 0$

$1 : bc = 0$

ii. $a = 0$ and $b = 0$ gives $f(x) = 0$, $x \neq -c$ (a special case of **a. ii.**)

$a = 0$ and $c = 0$ gives: $f(x) = \frac{b}{x}, x \neq 0$

23. **a.** Possible factors are $x \pm 1$, $x \pm 2$, $x \pm 3$

$p(-1) = -1+2+5-6 = 0$, so $x + 1$ is a factor

$p(2) = 8+8-10-6 = 0$, so $x - 2$ is a factor

Sum of roots is $-\frac{b}{a} = -2$, so 3rd root is -3, so $x + 3$ is a factor

(or using product of roots)

Factorises as $(x + 1)(x - 2)(x + 3)$

b. $2x^3 + 3x^2 - x - 2 \geq x^3 + x^2 + 4x + 4 \Rightarrow x^3 + 2x^2 - 5x - 6 \geq 0$

$(x + 1)(x - 2)(x + 3) \geq 0$

Intercepts are at $x = -3, -1, 2$

Sign of $p(x)$ is given by the table

x	$x < 3$	-3	$-3 < x < -1$	-1	$-1 < x < 2$	2	$x > 2$
$p(x)$	–ve	0	+ve	0	–ve	0	+ve

The inequality is true in the intervals $[-3, -1]$ and $[2, \infty[$

24. **a.** **i.** $\frac{dy}{dx} = 3ax^2 + 2bx + c$ **ii.** $\frac{d^2y}{dx^2} = 6ax + 2b$

b. **i.** sum of roots is $\frac{-2b}{3a}$ **ii.** sum of roots is $\frac{-2b}{6a} = \frac{-b}{3a}$

c. Let α and β be the x-values of the two turning points of y.

Then α and β are the roots of the quadratic equation $\frac{dy}{dx} = 0$

So $\alpha + \beta = \frac{-2b}{3a}$

Let X be the x-value of the point of inflection of y.

Then X is the root of the linear function $\frac{d^2y}{dx^2} = 0$

So $X = \dfrac{-b}{3a}$

Hence $\alpha + \beta = 2X \Rightarrow X = \dfrac{\alpha+\beta}{2}$, showing the desired result.

d. i. $p'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots + 2a_2 x + a_1$

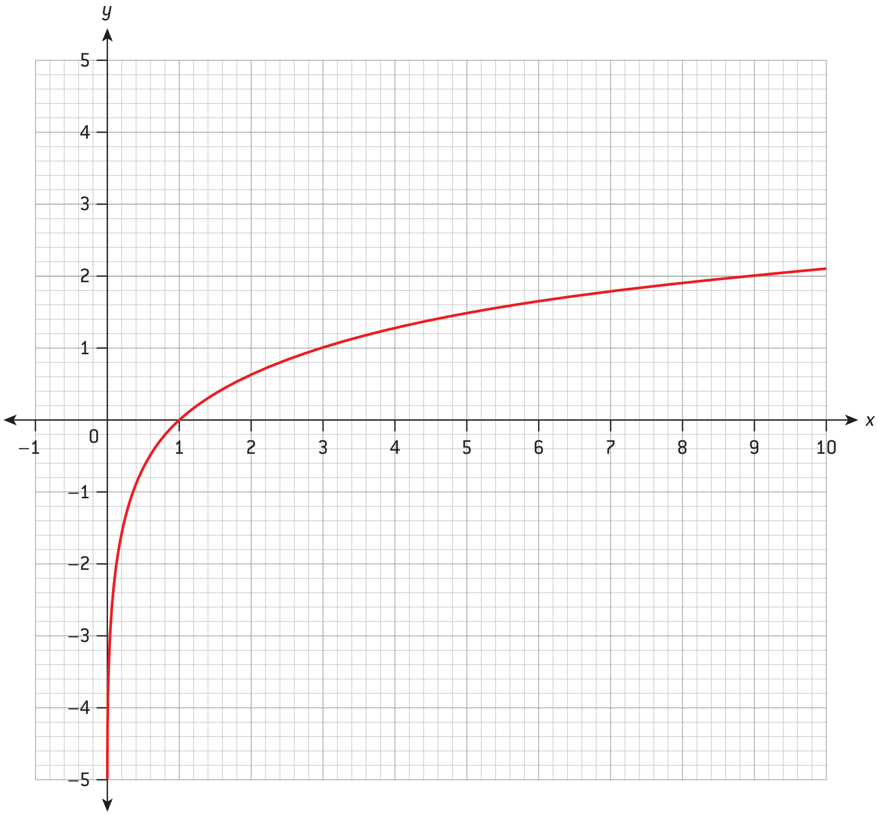
ii. $p''(x) = n(n-1)a_n x^{n-2} + (n-1)(n-2)a_{n-1}x^{n-3} + \ldots + 2a_2$

e. i. $S = \dfrac{-(n-1)a_{n-1}}{na_n}$

ii. $T = \dfrac{-(n-1)(n-2)a_{n-1}}{n(n-1)a_n} = \dfrac{-(n-2)a_{n-1}}{na_n}$

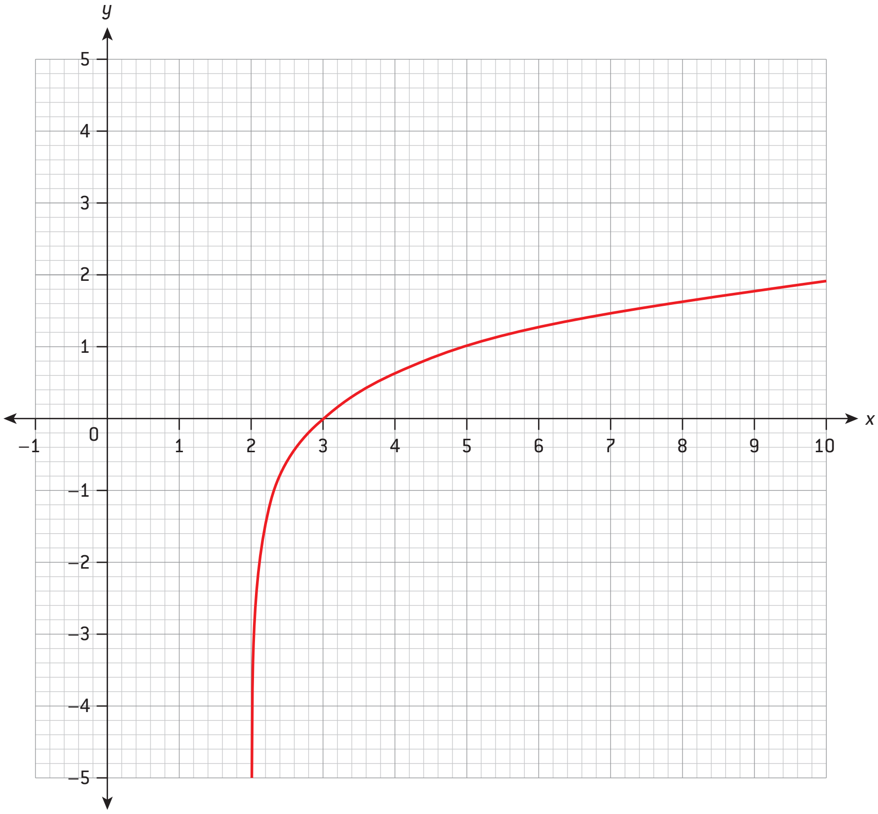
f. $T = \dfrac{-(n-1)(n-2)a_{n-1}}{n(n-1)a_n} = \dfrac{(n-2)}{(n-1)}S$, as required

25. a.



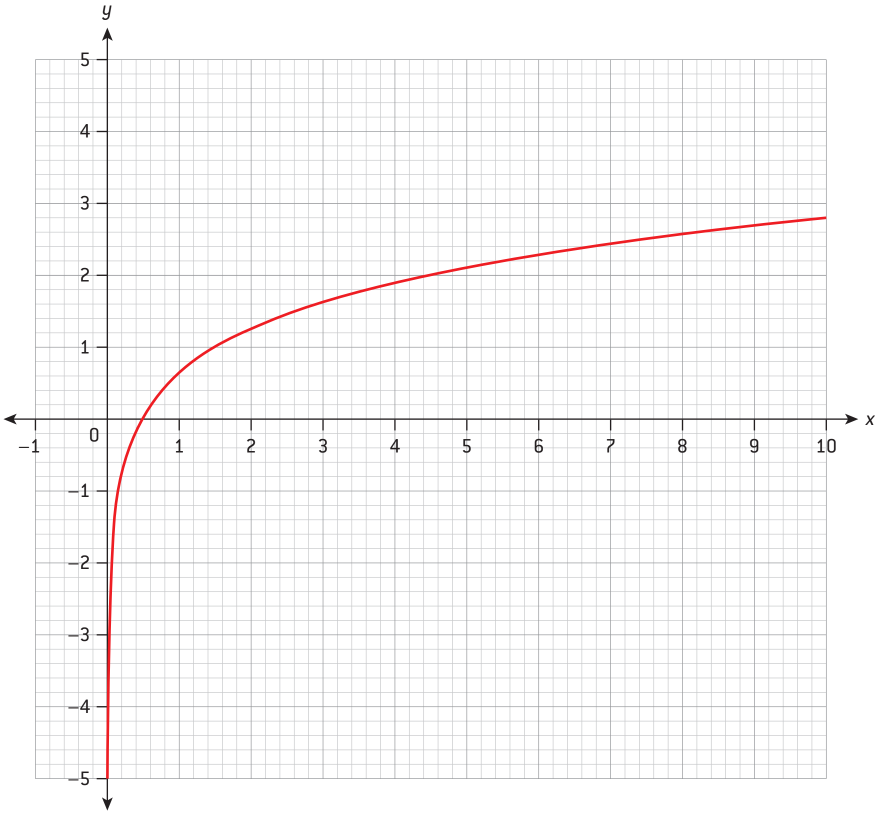
x-intercept is (1, 0)

b.



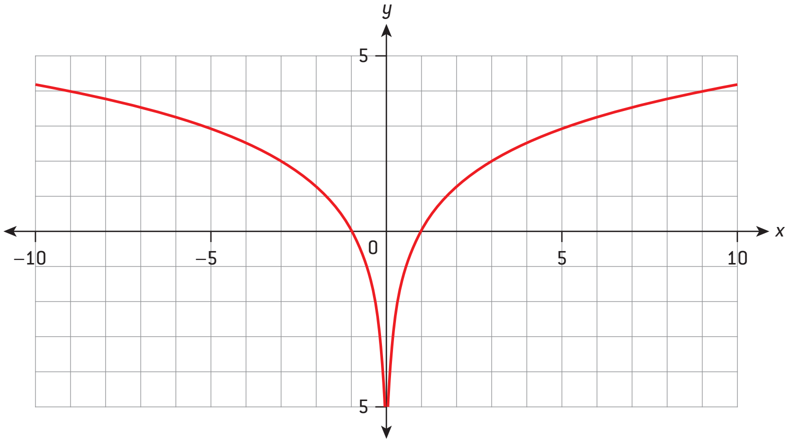
x-intercept is (3, 0)

c.



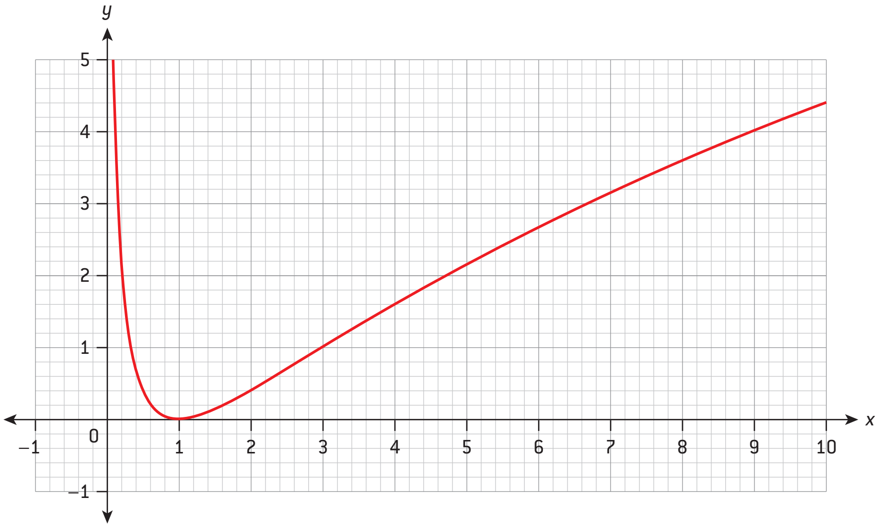
x-intercept is (0.5, 0)

d.



x-intercepts are (1, 0) and (−1, 0)

e.



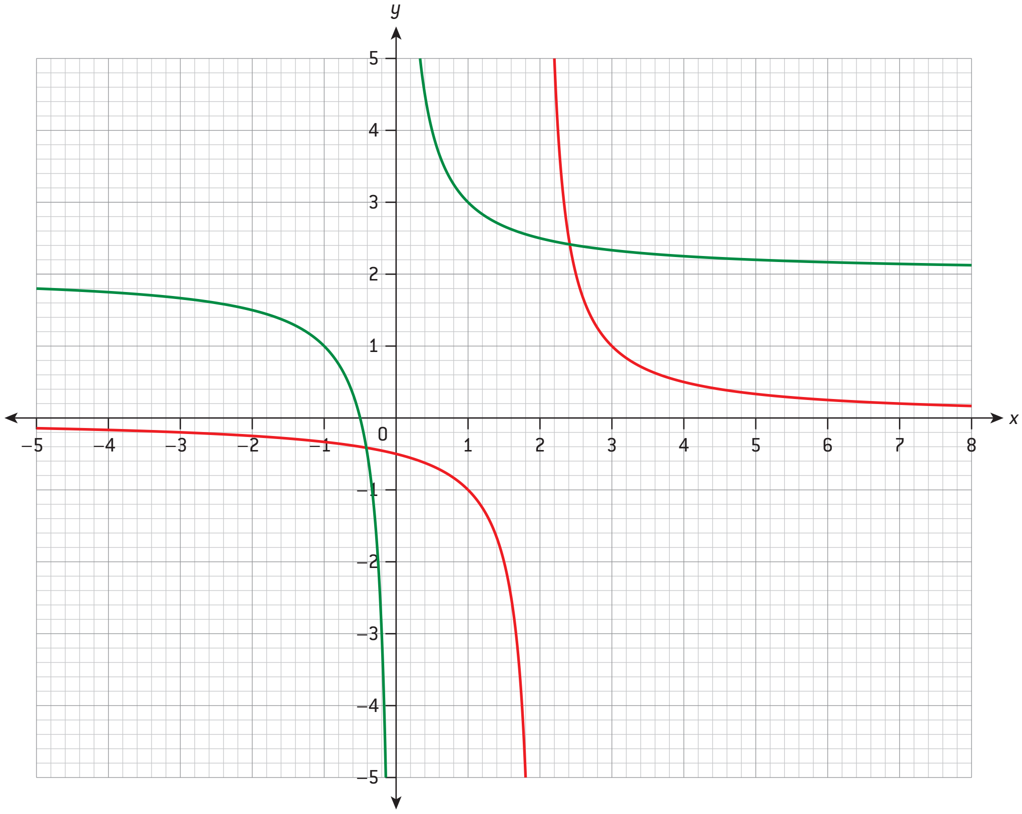
x-intercept is (1, 0)

26. a. $y=\dfrac{1}{x-2}$

Inverse given by $x=\dfrac{1}{y-2}\Rightarrow y-2=\dfrac{1}{x},\ y=\dfrac{1}{x}+2$

$f^{-1}(x)=\dfrac{1}{x}+2$, domain: $\{x\in\mathbb{R}|x\neq 0\}$, range $\{y\in\mathbb{R}|y\neq 2\}$

b. Sketching $f(x)$and $f^{-1}(x)$



Solving $f(x) = x$

(Note: you can solve $f(x)= f^{-1}(x)$ to find the intersection, but these curves are inverses, so they meet on the line $y = x$. Hence, it's easier to solve $f(x) = x$)

$$\frac{1}{x-2}=x \Rightarrow x^2-2x-1=0 \Rightarrow x=1\pm\sqrt{2}$$

$f(x)>f^{-1}(x)$ in the intervals $]1-\sqrt{2}, 0[$ and $]2, 1+\sqrt{2}[$

27. a. $x^2-4x+4=(x-2)^2$, so $x = 2$ is a vertical asymptote.

As $x \to \pm\infty$, the quadratic dominates the linear function so $y = 0$ is a horizontal asymptote.

$x \to 2^+ \Rightarrow y \Rightarrow +\infty, x \to 2^- \Rightarrow y \Rightarrow +\infty$

$x \to +\infty \Rightarrow y \Rightarrow 0^+, x \to -\infty \Rightarrow y \Rightarrow 0^-$

b. $x=0 \Rightarrow y=\frac{3}{4}$ so $\left(0, \frac{3}{4}\right)$ is the intercept on the y-axis

$y=0 \Rightarrow x=-3$ so $(-3, 0)$ is an intercept on the x-axis

c. $$\frac{dy}{dx}=\frac{1(x^2-4x+4)-(x+3)(2x-4)}{(x^2-4x+4)^2}=\frac{-(x-2)(x+8)}{(x-2)^4}=\frac{-(x+8)}{(x-2)^3}$$

$$\frac{dy}{dx}=0 \Rightarrow x=-8$$

Creating a sign table:

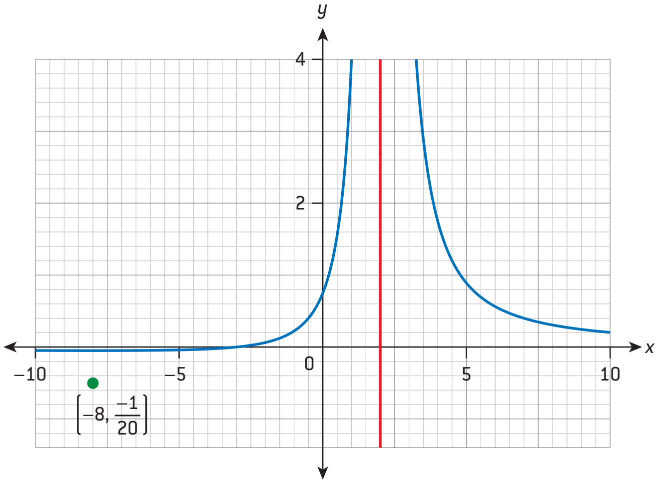
x	$x<-8$	-8	$-8<x<2$	2	$x>2$
$\frac{dy}{dx}$	–ve	0	+ve	undefined	–ve

So $\left(-8, \frac{-1}{20}\right)$ is a minimum

d. **i.** $]-8, 2[$

ii. $]-\infty, -8[\cup]2, \infty[$

e.



28. **a.** $x^2 + x + 4 = 0$ has a discriminant of $1 - 16 = -15$, so it has no roots and there are no vertical asymptotes

As $x \to \pm\infty$ the quadratic dominates the linear function so $y = 0$ is a horizontal asymptote.

$x \to +\infty \Rightarrow y \Rightarrow 0^+,\ x \to -\infty \Rightarrow y \Rightarrow 0^-$

b. $x = 0 \Rightarrow y = \frac{3}{4}$ so $\left(0, \frac{3}{4}\right)$ is the intercept on the y-axis

$y = 0 \Rightarrow x = -3$ so $(-3, 0)$ is an intercept on the x-axis

c. $\frac{dy}{dx} = \frac{1(x^2+x+4)-(x+3)(2x+1)}{(x^2+x+4)^2} = \frac{-x^2-6x+1}{(x^2+x+4)^2}$

$\frac{dy}{dx} = 0 \Rightarrow x^2 + 6x - 1 = 0,\ x = \frac{-6 \pm \sqrt{36+4}}{2} = -3 \pm \sqrt{10}$

So, there are stationary points when $x = -3 \pm \sqrt{10}$

Creating a sign table:

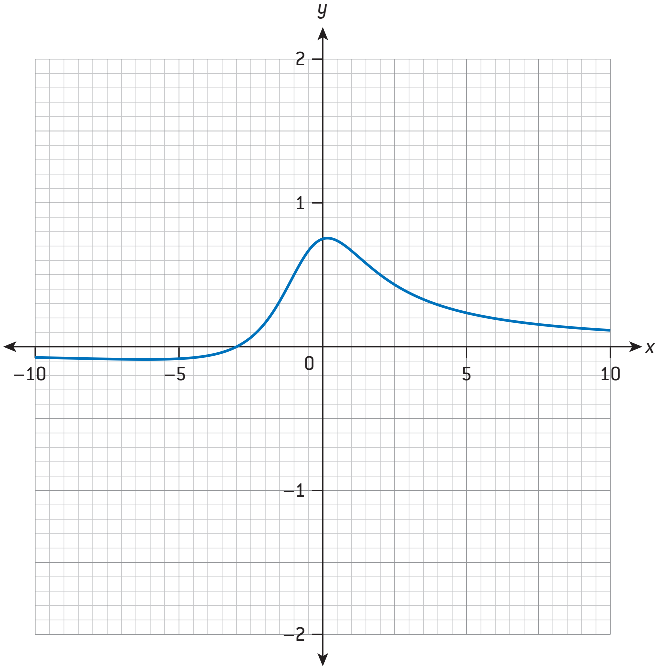
x	$x < -3-\sqrt{10}$	$-3-\sqrt{10}$	$-3-\sqrt{10} < x < -3+\sqrt{10}$	$-3+\sqrt{10}$	$x > -3+\sqrt{10}$
$\frac{dy}{dx}$	–ve	0	+ve	0	–ve

So, $x = -3-\sqrt{10}$ gives a minimum, $x = -3+\sqrt{10}$ gives a maximum

d. **i.** $]-3-\sqrt{10}, -3+\sqrt{10}[$

ii. $]-\infty, -3-\sqrt{10}[\cup]-3+\sqrt{10}, \infty[$

e.



29. a. $x + 3 = 0 \Rightarrow x = -3$, so $x = -3$ is a vertical asymptote

The quadratic dominates the linear function as x becomes large, so $f(x) \to \pm\infty$, as $x \to \pm\infty$. Hence no horizontal asymptotes.

$$\begin{array}{r|l} & x-7 \\ \hline x+3 & x^2-4x+4 \\ & \underline{x^2+3x} \\ & \quad -7x+4 \\ & \quad \underline{-7x-21} \\ & \qquad 25 \end{array}$$

$\frac{x^2-4x+4}{x+3} = x-7+\frac{25}{x+3}$, so $y = x - 7$ is an oblique asymptote.

b. $x = 0 \Rightarrow y = \frac{4}{3}$, so intercept on y-axis at $\left(0, \frac{4}{3}\right)$

$x^2 - 4x + 4 = (x-2)^2 = 0 \Rightarrow x = -2$ so intercept on x-axis at $(-2, 0)$

c. $f(x) = \frac{(x-2)^2}{x+3} \Rightarrow f'(x) = \frac{2(x-2)(x+3)-(x-2)^2 \times 1}{(x+3)^2} = \frac{(x-2)(x+8)}{(x+3)^2}$

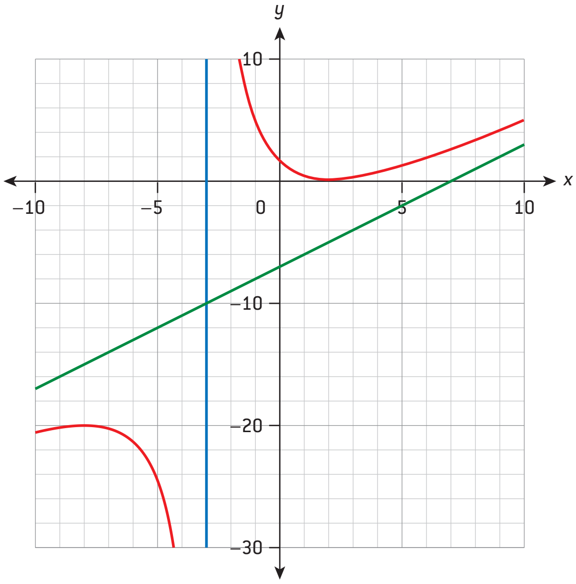
$f'(x) = 0 \Rightarrow x = 2 \text{ or } -8$

Using a sign table:

x	$x < -8$	-8	$-8 < x < -3$	-3	$-3 < x < 2$	2	$x > 2$
$f'(x)$	+ve	0	−ve	undefined	−ve	0	+ve

So, maximum at $(-8, -20)$, minimum at $(2, 0)$ (expected turning point due to double root)

d.



e. **i.** Maximum domain is $\{x \in \mathbb{R} | x \neq -3\}$

ii. Corresponding range is $]-\infty, -20] \cup [0, +\infty[$

30. **a.** $2x + 6 = 0 \Rightarrow x = -3$ so $x = -3$ is a vertical asymptote.

The quadratic dominates the linear function as x becomes large, so $f(x) \to \pm\infty$ as $x \to \pm\infty$. Hence, no horizontal asymptotes.

$$\begin{array}{r|l} & \frac{1}{2}x - 1 \\ \hline 2x+6 & x^2 + x - 12 \\ & x^2 + 3x \\ \hline & -2x - 12 \\ & -2x - 6 \\ \hline & -6 \end{array}$$

$\dfrac{x^2+x-12}{2x+6} = \dfrac{1}{2}x - 1 + \dfrac{-6}{2x+6}$ so $y = \dfrac{1}{2}x - 1$ is an oblique asymptote.

b. $x = 0 \Rightarrow y = -2$, so intercept on y-axis at $(0, -2)$

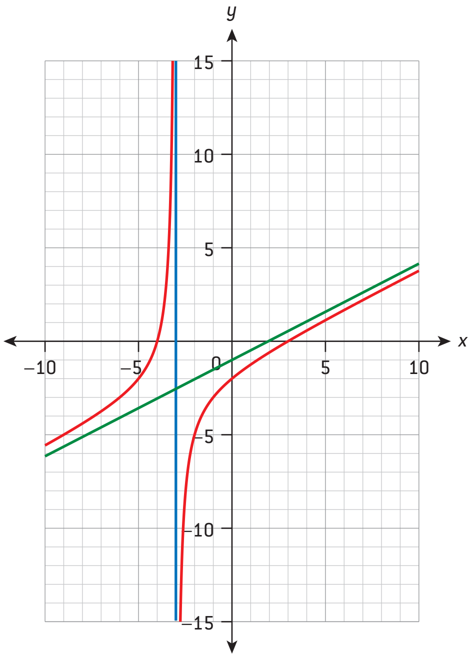
$x^2 + x - 12 = (x-3)(x+4) = 0 \Rightarrow x = 3$ or -4, so intercepts on x-axis at $(-4, 0)$ and $(3, 0)$

c. $f(x) = \dfrac{x^2+x-12}{2x+6} \Rightarrow f'(x) = \dfrac{(2x+1)(2x+6) - (x^2+x-12)\times 2}{(2x+6)^2}$

$$= \frac{2x^2+12x+30}{(2x+6)^2}$$

Discriminant of $2x^2 + 12x + 30$ is negative so there are no turning points. $f'(x)$ is always positive so graph is always increasing.

d.



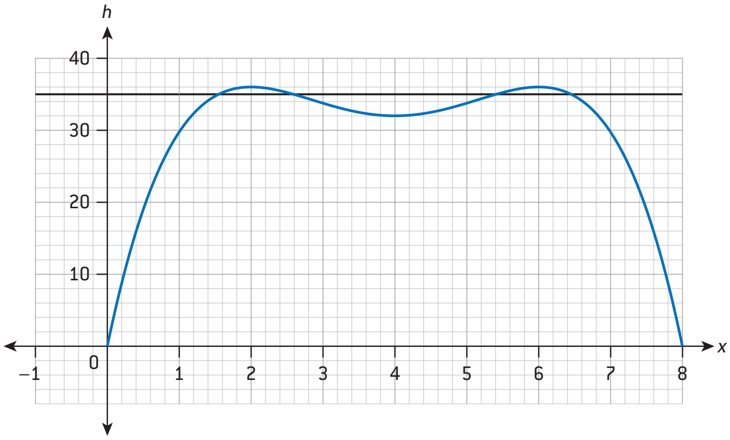
e. **i.** maximum domain is $\{x \in \mathbb{R} | x \neq -3\}$

ii. corresponding range is $\{y \in \mathbb{R}\}$

Link between examples in the chapter and problems on rational functions and the graphs of reciprocal functions.

HL Paper 2: Section A
Technology required

31. **a.** Considering the graph:



They will be walking in snow for x in the intervals [1.39, 2.92] and [5.08, 6.61]

b. They will be walking in snow for a horizontal distance of $1.53 \times 2 \times 5 = 15.3\,\text{km}$ (3sf)

c. $h(0) = h(8) = 0$ Both huts are at the same height.

HL Paper 2: Section B
Technology required

32. a. $p(2) = 8a + 4b + 4 + 1 = 49 \Rightarrow 8a + 4b = 44$

$p(-1) = -a + b - 2 + 1 = -2 \Rightarrow -a + b = -1$

Solving $a = 4$, $b = 3$

b. $p(1) = 4 + 3 + 2 + 1 = 10$

HL Paper 3
Technology required

33. a. i. $g^2(x) = g(2x+1) = 4x+3$

ii. $g^3(x) = g(4x+3) = 8x+7$

iii. $g^4(x) = g(8x+7) = 16x+15$

b. Conjecture that $g^n(x) = 2^n x + (2^n - 1)$

c. $y = 2x + 1$

Inverse given by $x = 2y+1 \Rightarrow 2y = x-1 \Rightarrow y = \frac{1}{2}x - \frac{1}{2}$

So $g^{-1}(x) = \frac{1}{2}x - \frac{1}{2}$

d. Using $y = 2^n x + (2^n - 1)$, the inverse is given by

$x = 2^n y + (2^n - 1) \Rightarrow 2^n y = x - (2^n - 1)$

$$y = \frac{1}{2^n}x - \frac{(2^n - 1)}{2^n}$$

Conjecture that $(g^n)^{-1}(x) = \frac{1}{2^n}x - \frac{(2^n - 1)}{2^n}$

e. $g^2(x) = 4x+3; [g(x)]^2 = (2x+1)^2 = 4x^2 + 4x + 1$

so $g^2(x) \neq [g(x)]^2$

f. Let $P(n)$ be the statement that $f^n(x) = a^n x + \frac{b(a^n - 1)}{a-1}$

$f^1(x) = ax + b$ so $P(1)$ is true.

Assume $P(k)$ is true, that is: $f^k(x) = a^k x + \frac{b(a^k - 1)}{a-1}$

then $f^{k+1}(x) = f\left(a^k x + \frac{b(a^k - 1)}{a-1}\right) = a\left(a^k x + \frac{b(a^k - 1)}{a-1}\right) + b$

$$a^{k+1}x + a\frac{b(a^k - 1)}{a-1} + b = a^{k+1}x + \frac{ba^{k+1} - ab + ab - b}{a-1} = a^{k+1}x + \frac{b(a^{k+1} - 1)}{a-1}$$

Showing that $P(k + 1)$ is true

Since $P(1)$ is true and $P(k)$ true $\Rightarrow P(k + 1)$ true, then by the principle of mathematical induction, it has been proved that $P(n)$ is true for all $n \in \mathbb{Z}^+$.

g. Using $y = a^n x + \frac{b(a^n - 1)}{a - 1}$, the inverse is given by

$$x = a^n y + \frac{b(a^n - 1)}{a - 1} \Rightarrow a^n y = x - \frac{b(a^n - 1)}{a - 1}$$

$$y = \frac{1}{a^n}x - \frac{b(a^n - 1)}{a^n(a - 1)} \qquad \text{so } (g^n)^{-1}(x) = \frac{1}{a^n}x - \frac{b(a^n - 1)}{a^n(a - 1)}$$

h. i. If he had amount X in his account during one year then during the next year he will have $1.05X + 100$.

So, the function $f(x) = ax + b$ with $a = 1.05$, $b = 100$ gives the amount in his account after he pays in £100 on the following 1 January.

Hence $f^{10}(x)$ gives the amount in his account after he pays in £100 on 1 January 2010.

So, the formula in part **f.** can be used with $a = 1.05$, $b = 100$, and $n = 10$

ii. $1000(1.05)^{10} + 100\frac{(1.05^{10} - 1)}{0.05} = £2886.68 \text{ (2 dp)}$

i. Using the formula from part **g.** with $a = 1.05$, $b = 300$, and $n = 10$ gives

$$Y = \frac{6000}{1.05^{10}} - 300\frac{(1.05^{10} - 1)}{1.05^{10}(1.05 - 1)} = £1366.96 \text{ (2 dp)}$$

3 GEOMETRY AND TRIGONOMETRY: END-OF-CHAPTER PRACTICE QUESTION WORKED SOLUTIONS

SL Paper 1: Section A
No technology allowed

1. **a.** $\tan\theta = \frac{2}{6} = \frac{1}{3}$

b. Gradient is $\frac{1}{3}$; $\frac{b}{10} = \frac{1}{3} \Rightarrow b = \frac{10}{3}$

c. $\frac{3}{c} = \frac{1}{3} \Rightarrow c = 9$

2. **a.** $\sin 30° = \frac{\text{rise}}{600}$

Vertical height $= 600 \times \frac{1}{2} = 300$ m

b. Inclined length is hypotenuse h:

$\frac{600}{h} = \cos 30 = \frac{\sqrt{3}}{2} \Rightarrow h = \frac{1200}{\sqrt{3}} = \frac{1200\sqrt{3}}{3} = 400\sqrt{3}$

So $k = 400$

c. Angle of depression is also 30°

3. $s = r\theta = 12$

$A = \frac{1}{2}r^2\theta = 360$

$360 = \frac{r}{2} \times 12 \Rightarrow r = 60$

$\theta = \frac{12}{60} = \frac{1}{5}$

4. $(\sin x + \cos x)^2 = 1 + 2\cos x$

$\sin^2 x + 2\sin x\cos x + \cos^2 x = 1 + 2\cos x$

$2\sin x \cos x = 2\cos x$

$2\sin x \cos x - 2\cos x = 0$

$2\cos x\,(1 - \sin x) = 0$

$\cos x = 0$ or $\sin x = 1$

$x = \frac{\pi}{2}, x = \frac{3\pi}{2}$ or $x = \frac{5\pi}{2}$

5. $6 - \cos 2\theta + 7\sin\theta = 0$

$6 - (1 - 2\sin^2\theta) + 7\sin\theta = 0$

$2\sin^2\theta + 7\sin\theta + 5 = 0$

$(2\sin\theta + 5)(\sin\theta + 1) = 0$

$\sin\theta = -1$, rejecting $\sin\theta = -\frac{5}{2}$

$\theta = \frac{3\pi}{2}$

6. **a.** In the interval $\left[0, \frac{\pi}{2}\right]$, sine is an increasing function so

$A > B > C \Rightarrow \sin A > \sin B > \sin C$

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{a}{b} = \frac{\sin A}{\sin B}$ and $\frac{b}{c} = \frac{\sin B}{\sin C}$

Giving the desired result that $a > b > c$

b. The result is still true. $A + B + C = \pi \Rightarrow \pi - A = B + C$

So $\pi - A > B > C \Rightarrow \sin(\pi - A) > \sin B > \sin C$

Also $\sin(\pi - A) = \sin A$ so the proof now follows the last lines of that in part **a.**

7. **a.** Applying the cosine rule twice

$b^2 = AD^2 + \left(\frac{a}{2}\right)^2 - 2AD\frac{a}{2}\cos A\hat{D}C$

$c^2 = AD^2 + \left(\frac{a}{2}\right)^2 - 2AD\frac{a}{2}\cos A\hat{D}B$

$A\hat{D}B = 180° - A\hat{D}C \Rightarrow \cos A\hat{D}B = -\cos A\hat{D}C$

Adding the first two equations: $b^2 + c^2 = 2AD^2 + \frac{a^2}{2}$ giving

$4AD^2 = 2b^2 + 2c^2 - a^2$, as required.

b. Smallest median of length l will have $4l^2 = 2\times 3^2 + 2\times 4^2 - 5^2 = 25$

So $l = \frac{5}{2}$

8. **a. i.** $\frac{1}{2}l^2\theta$ **ii.** $l\theta$

b. Curved area of cone will also be $\frac{1}{2}l^2\theta$

Arc length will become the circumference of the circular base. So $l\theta = 2\pi r$

Area $= \frac{1}{2}l^2\theta = \frac{1}{2}l(l\theta) = \frac{1}{2}l2\pi r = \pi rl$, as required.

SL Paper 1: Section B
No technology allowed

9. **a.** $P = (\cos B, \sin B), Q = (\cos A, \sin A)$

b. $PQ = \sqrt{(\sin^2 A - \sin^2 B) + (\cos^2 A - \cos^2 B)}$

$= \sqrt{\sin^2 A - 2\sin A\sin B + \sin^2 B + \cos^2 A - 2\cos A\cos B + \cos^2 B}$

$= \sqrt{2 - 2\sin A\sin B - 2\cos A\cos B}$

c. $PQ^2 = 1^2 + 1^2 - 2\times 1\times 1\cos(A - B) = 2 - 2\cos(A - B)$

d. Equating PQ^2, we have $2 - 2\sin A\sin B - 2\cos A\cos B = 2 - 2\cos(A - B)$

Giving $\cos(A - B) = \cos A\cos B + \sin A\sin B$, as required

SL Paper: 2 Section A
Technology required

10. a. $\frac{h}{15} = \tan 55 \Rightarrow h = 15\tan 55 = 21.4222... = 21.4$ m (3 sf)

b. The angle of depression from top of the tree to Pam's feet (call it α) equals the angle of elevation from Pam's feet to the top of the tree.

$\frac{h}{20} = \tan\alpha \Rightarrow \alpha = \arctan\frac{21.4222...}{20} = 47.0°$ (3 sf)

11. a. $M = \left(\frac{1+3}{2}, \frac{2+4}{2}, \frac{3+7}{2}\right) = (2, 3, 5)$

b. $CD = \sqrt{(3-1)^2 + (4-1)^2 + (7-1)^2} = \sqrt{4+9+36} = \sqrt{49} = 7$

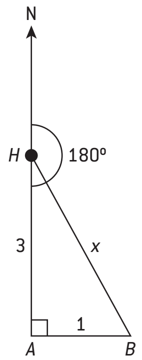
c. $AC = \sqrt{3^2 + 4^2 + 7^2} = \sqrt{9+16+49} = \sqrt{74}$

$AD = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

$7^2 = 74 + 3 - 2\times\sqrt{74}\sqrt{3}\cos A \Rightarrow \cos A = \frac{28}{2\times\sqrt{74}\sqrt{3}}$

$C\hat{A}D = 20.0°$ (3 sf)

12. a.



b. $\sqrt{3^2 + 1^2} = \sqrt{10} = 3.16$ km (3 sf)

c. Bearing will be $180 - \arctan\frac{1}{3} = 162°$ (to nearest degree)

13. a. Volume of cuboid $= 2\times3\times4 = 24\,\text{m}^3$ Volume of sphere $= \frac{4}{3}\pi\times1^3$

$\frac{24}{\left(\frac{4\pi}{3}\right)} = 5.729...$ So maximum number of spheres is 5.

b. $24 - 5\times\frac{4\pi}{3} = 3.0560... = 3.06\,\text{m}^3$ (3 sf)

c. $x^3 = 3.0560... \Rightarrow x = 1.45$ m (3 sf)

14. a. $\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2(2r) = 100 \Rightarrow r^3 = \frac{150}{\pi}$ $r = 3.62783... = 3.63$ cm (3 sf)

b. $l = \sqrt{r^2 + (2r)^2} = \sqrt{5}r$

Surface area $= \frac{1}{3}\pi rl + \pi r^2$

Surface area $= \frac{1}{3}\pi\sqrt{5}(3.62783...)^2 + \pi(3.62783...)^2 = 72.2\,\text{cm}^2$ (3 sf)

15. a. Area of triangle $POQ = \frac{1}{2}OP \times OQ \times \sin POQ = \frac{1}{2} \times 10^2 \times \sin 69° = 46.679\ldots$

Area of sector $= \pi \times 10^2 \times \frac{69}{360} = 60.213\ldots$

Area between arc and chord $= 60.213\ldots - 46.679\ldots = 13.5\,\text{cm}^2$ (3 sf)

b. $\frac{13.534\ldots}{\pi \times 10^2} \times 100\% = 4.31\%$(3 sf)

SL Paper 2: Section B
Technology required

16. Let $A\hat{O}P = \alpha \qquad \cos\alpha = \frac{1}{5}$

Length of string equals major arc AB plus 2 times distance AP

Angle at centre of major arc is $2\pi - 2\alpha \qquad AP = \sqrt{5^2 - 1^2} = \sqrt{24}$

Length is $1 \times \left(2\pi - 2\arccos\frac{1}{5}\right) + 2\sqrt{24} = 13.3$ m (3 sf)

HL Paper 1: Section A
No technology allowed

17. a. $\tan 4\theta = \frac{\sin 2(2\theta)}{\cos 2(2\theta)} = \frac{2\sin 2\theta \cos 2\theta}{1 - 2\sin^2 2\theta} = \frac{4\sin\theta\cos\theta(1 - 2\sin^2\theta)}{1 - 8\sin^2\theta\cos^2\theta}$

$= \frac{4\sin\theta\cos\theta - 8\sin^3\theta\cos\theta}{1 - 8\sin^2\theta\cos^2\theta}$, as required

b. $4\sin\theta\cos\theta - 8\sin^3\theta\cos\theta = 1 - 8\sin^2\theta\cos^2\theta \Rightarrow \tan 4\theta = 1$

$4\theta = \frac{\pi}{4}, \frac{5\pi}{4}$

$\theta = \frac{\pi}{16}, \frac{5\pi}{16}$

18. $\cos\left(x + \frac{\pi}{6}\right) = \cos x \cos\frac{\pi}{6} - \sin x \sin\frac{\pi}{6} = \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x$

$\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x = 5\cos x$

$\sin x = (\sqrt{3} - 10)\cos x$

$\tan x = \sqrt{3} - 10$

19. a. Area of $AOP = \frac{1}{2}r^2 \sin\theta$

b. $TP = r \tan \theta$

Area of $POT = \frac{1}{2}r(r \tan\theta) = \frac{1}{2}r^2 \tan\theta$

c. Area of sector $OAP = \frac{1}{2}r^2\theta$

d. Area of triangle OAP < area of sector OAP < area of triangle POT

$$\frac{1}{2}r^2\sin\theta < \frac{1}{2}r^2\theta < \frac{1}{2}r^2\tan\theta$$

$\therefore \sin\theta < \theta < \tan\theta$

20. a. $\tan(A+B) \equiv \dfrac{\sin(A+B)}{\cos(A+B)} \equiv \dfrac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$, dividing numerator and denominator by $\cos A\cos B$:

$$\equiv \frac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{1 - \dfrac{\sin A\sin B}{\cos A\cos B}} \equiv \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

b. $\tan 75° = \tan(45° + 30°) = \dfrac{\tan 45° + \tan 30°}{1 - \tan 45°\tan 30°}$

$$= \frac{1 + \dfrac{1}{\sqrt{3}}}{1 - \dfrac{1}{\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1} \qquad \left(= \frac{(\sqrt{3}+1)^2}{2} = 2 + \sqrt{3} \right)$$

21. a. $\overrightarrow{AP} = -\mathbf{a} + \mathbf{p}$, $\overrightarrow{PB} = -\mathbf{p} - \mathbf{a}$

b. $\overrightarrow{AP} \bullet \overrightarrow{PB} = (-\mathbf{a} + \mathbf{p}) \bullet (-\mathbf{p} - \mathbf{a}) = \mathbf{a} \bullet \mathbf{a} - \mathbf{p} \bullet \mathbf{p}$ since $\mathbf{a} \bullet \mathbf{p} = \mathbf{p} \bullet \mathbf{a}$

$\overrightarrow{AP} \bullet \overrightarrow{PB} = r^2 - r^2 = 0$ showing that the vectors are perpendicular and hence proving the result.

HL Paper 1: Section B
No technology allowed

22. a. $\mathbf{c} = \mathbf{b} + \mathbf{a}$

b. $\mathbf{c} \bullet \mathbf{c} = (\mathbf{a} + \mathbf{b}) \bullet (\mathbf{a} + \mathbf{b}) = \mathbf{a} \bullet \mathbf{a} + \mathbf{a} \bullet \mathbf{b} + \mathbf{b} \bullet \mathbf{a} + \mathbf{b} \bullet \mathbf{b}$

Now $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a} = 0$ as these vectors are perpendicular, hence

$|\mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2$ as required.

c. $\mathbf{c} \bullet \mathbf{c} = (\mathbf{a} + \mathbf{b}) \bullet (\mathbf{a} + \mathbf{b}) = \mathbf{a} \bullet \mathbf{a} + \mathbf{a} \bullet \mathbf{b} + \mathbf{b} \bullet \mathbf{a} + \mathbf{b} \bullet \mathbf{b}$

$|\mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2|\mathbf{a}||\mathbf{b}|\cos(180 - C)$

$= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos C$, as required

23. a. $s = 16$ Area $= \sqrt{16 \times 12 \times 3 \times 1} = 24$

b. Area $= \dfrac{1}{2}ab\sin C = \dfrac{ab}{2}\sqrt{1 - \cos^2 C} = \dfrac{2ab}{4}\sqrt{1 - \left(\dfrac{a^2 + b^2 - c^2}{2ab}\right)^2}$

$$= \frac{1}{4}\sqrt{4a^2b^2\left(1 - \left(\frac{a^2 + b^2 - c^2}{2ab}\right)^2\right)}$$

Giving: Area $= \dfrac{1}{4}\sqrt{4a^2b^2 - (a^2 + b^2 - c^2)^2}$ as required.

c. $\frac{1}{4}\sqrt{4a^2b^2-(a^2+b^2-c^2)^2}=\frac{1}{4}\sqrt{(2ab-(a^2+b^2-c^2))(2ab+(a^2+b^2-c^2))}$

$=\frac{1}{4}\sqrt{(c^2-(a^2-2ab+b^2))((a^2+2ab+b^2)-c^2))}$

$=\frac{1}{4}\sqrt{(c^2-(a-b)^2)((a+b)^2-c^2)}=\frac{1}{4}\sqrt{(c-(a-b))(c+(a-b))((a+b)-c)((a+b)+c)}$

$=\sqrt{\left(\frac{c-a+b}{2}\right)\left(\frac{c+a-b}{2}\right)\left(\frac{a+b-c}{2}\right)\left(\frac{a+b+c}{2}\right)}$

$=\sqrt{\left(\frac{a+b+c-2a}{2}\right)\left(\frac{a+b+c-2b}{2}\right)\left(\frac{a+b+c-2c}{2}\right)\left(\frac{a+b+c}{2}\right)}$

$=\sqrt{(s-a)(s-b)(s-c)s}$ as required, completing the proof.

24. a. $\mathbf{v}\cdot\mathbf{w}=\cos^2\alpha-1+\sin\alpha$

b. $\cos^2\alpha-1+\sin\alpha=0\Rightarrow-\sin^2\alpha+\sin\alpha=0\Rightarrow\sin\alpha(1-\sin\alpha)=0$

$\sin\alpha=0$ or 1 $\quad\alpha=0,\pi,2\pi$ or $\frac{\pi}{2}$

c. $\mathbf{v}\times\mathbf{w}=(1+\sin\alpha)\mathbf{i}+(\cos\alpha\sin\alpha-\cos\alpha)\mathbf{j}-2\cos\alpha\,\mathbf{k}$

d. $\mathbf{v}\times\mathbf{w}=\mathbf{0}$ requires $\cos\alpha=0$ and $\sin\alpha=-1$ so $\alpha=\frac{3\pi}{2}$

25. a. No solution since $1+\tan^2 x=\sec^2 x$ and $2\sec^2 x=\frac{2}{\cos^2 x}=0$ is impossible

b. No solution since $\sin x\cos x=\frac{1}{2}\sin 2x$ so the equation implies that $\sin 2x=2$ but $-1\le\sin 2x\le 1$

c. $4\cos x\sin^2 x-\cos x=0\Rightarrow\cos x(4\sin^2 x-1)=0$

$\cos x=0$ or $\sin x=\pm\frac{1}{2}$

$x=30°,90°,150°,210°,270°,330°$

d. No solution since $-1\le\sin x\le 1$ and $-1\le\cos x\le 1$ so $-7\le 3\sin x+4\cos x\le 7$

26. a. $A\hat{B}X=\theta-\alpha, A\hat{C}X=180°-\theta-\beta$ $\qquad$ Applying the sine rule twice:

$\frac{ma}{\sin\alpha}=\frac{AX}{\sin(\theta-\alpha)}$

$\frac{na}{\sin\beta}=\frac{AX}{\sin(180°-\theta-\beta)}=\frac{AX}{\sin(\theta+\beta)}$

$\frac{AX}{a}=\frac{m\sin(\theta-\alpha)}{\sin\alpha}=\frac{n\sin(\theta+\beta)}{\sin\beta}$, as required

b. $\frac{m(\sin\theta\cos\alpha-\sin\alpha\cos\theta)}{\sin\alpha}=\frac{n(\sin\theta\cos\beta+\sin\beta\cos\theta)}{\sin\beta}$

$m\sin\theta\cot\alpha-m\cos\theta=n\sin\theta\cot\beta+n\cos\theta$

Using $m+n=1$ and dividing through by $\sin\theta$

$m\cot\alpha=n\cot\beta+\cot\theta$

So $\cot\theta=m\cot\alpha-n\cot\beta$, as required

c. Putting $m=n=\frac{1}{2}$ gives $2\cot\theta=\cot\alpha-\cot\beta$, as required.

27. **a.** $\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\dfrac{x+y}{1-xy}$

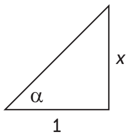
So $\alpha+\beta=\arctan\dfrac{x+y}{1-xy}$ giving:

$\arctan x+\arctan y=\arctan\left(\dfrac{x+y}{1-xy}\right)$, as required.

b. $\dfrac{x+y}{1-xy}$ is undefined if the denominator is zero

So undefined if $xy=1$

c.



$\tan\alpha=\dfrac{x}{1}$ so $\tan\left(\dfrac{\pi}{2}-\alpha\right)=\dfrac{1}{x}$

$\dfrac{\pi}{2}-\alpha=\arctan\dfrac{1}{x}\Rightarrow\dfrac{\pi}{2}-\arctan x=\arctan\dfrac{1}{x}$, giving:

$\arctan x+\arctan\dfrac{1}{x}=\dfrac{\pi}{2}$, as required

d. Part **a.** could not be used because $x\times\dfrac{1}{x}=1$

28. **a.** $\begin{pmatrix}2\\-1\\1\end{pmatrix}$ is not a multiple of $\begin{pmatrix}1\\1\\2\end{pmatrix}$, so they are not parallel.

To intersect, require: $\begin{cases}1+2\lambda=\mu\\1-\lambda=-5+\mu\\1+\lambda=4+2\mu\end{cases}$

Adding last 2 equations: $2=-1+3\mu\Rightarrow\mu=1$

Putting this in the first equation gives $\lambda=0$, which then gives a contradiction.

So the lines are skew.

b. Reading off the second line is parallel to $\begin{pmatrix}6\\-2\\2\end{pmatrix}$ which is a multiple of $\begin{pmatrix}-3\\1\\-1\end{pmatrix}$

So the lines are parallel.

$(4, 0, -6)$ is a point on the second line but it is not on the first line so they do not represent the same line.

c. $\begin{pmatrix}1\\-1\\2\end{pmatrix}$ is not a multiple of $\begin{pmatrix}2\\0\\1\end{pmatrix}$, so they are not parallel.

To intersect, requires: $\begin{cases}\lambda=1+2\mu\\-\lambda=1\\1+2\lambda=\mu\end{cases}$ so $\lambda=-1$ and $\mu=-1$

The lines intersect at $(-1, 1, -1)$

29. a. LHS $= \dfrac{1+2\sin A\cos A-(\cos^2 A-\sin^2 A)}{1+2\sin A\cos A+(\cos^2 A-\sin^2 A)} = \dfrac{2\sin A\cos A+2\sin^2 A}{2\sin A\cos A+2\cos^2 A}$

$= \dfrac{\sin A(\cos A+\sin A)}{\cos A(\sin A+\cos A)} = \dfrac{\sin A}{\cos A}$, since $\cos A+\sin A \neq 0$ for $A \in \left[0, \dfrac{\pi}{2}\right[$

Giving $\tan A$, as required.

b. i. $A+B=180-C \Rightarrow \sin(A+B)=\sin(180-C)=\sin C$

Giving $\sin A\cos B+\cos A\sin B=\sin C$ using the compound angle formula

ii. $\sin 90 = 1$

As $B=90-A$, then $\cos B = \sin A$ and $\sin B = \cos A$

Substituting $\sin A\sin A+\cos A\cos A=1$, so $\sin^2 A+\cos^2 A=1$

HL Paper 2: Section A
Technology required

30. Rewriting the second line as $\dfrac{x-0}{1}=\dfrac{y-1}{-2}=\dfrac{z+5}{2}$

consider a vector parallel to each line,

$$\mathbf{u}=\begin{pmatrix}2\\3\\4\end{pmatrix} \qquad \mathbf{v}=\begin{pmatrix}1\\-2\\2\end{pmatrix}$$

Let θ be the angle between the lines. Then $\cos\theta = \dfrac{|\mathbf{u}\bullet\mathbf{v}|}{|\mathbf{u}||\mathbf{v}|} = \dfrac{|2-6+8|}{\sqrt{29}\sqrt{9}} = \dfrac{4}{3\sqrt{29}}$

so $\theta = 75.7°$ (3 sf)

HL Paper 2: Section B
Technology required

31. a. i. $\mathbf{r}=\begin{pmatrix}0\\10\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\end{pmatrix}$

ii. $\mathbf{s}=\begin{pmatrix}0\\-5\end{pmatrix}+\mu\begin{pmatrix}2\\1\end{pmatrix}$

b. $\begin{pmatrix}0\\10\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}0\\-5\end{pmatrix}+\mu\begin{pmatrix}2\\1\end{pmatrix} \Rightarrow \lambda=2\mu,\ 15-\lambda=\mu\ ,15=3\mu, \mu=5$

Point is (10, 0)

c. $\mathbf{r}_A=\begin{pmatrix}0\\10\end{pmatrix}+t\begin{pmatrix}1\\-1\end{pmatrix} \qquad \mathbf{r}_B=\begin{pmatrix}0\\-5\end{pmatrix}+t\begin{pmatrix}2\\1\end{pmatrix}$

$\mathbf{r}_A=\mathbf{r}_B$ implies $t=2t$, $15-t=t \Rightarrow 15=0$ giving a contradiction, showing that they do not collide.

d. **i.** Distance between cars is $|\mathbf{r}_A - \mathbf{r}_B| = \left|\begin{pmatrix}0\\15\end{pmatrix} + t\begin{pmatrix}-1\\-2\end{pmatrix}\right| = \sqrt{t^2 + (15-2t)^2}$

So we need to minimise the quadratic $5t^2 - 60t + 225$. By the axis of symmetry of a concave up quadratic this happens at $t = 6$

ii. Shortest distance between cars is $\sqrt{36+9} = \sqrt{45} = 6.71\,\text{m}$ (3 sf)

iii. Ali's car is at position (6, 4) Ben's car is at position (12, 1)

32. **a.** $\begin{pmatrix}4\\-1\\3\end{pmatrix} \bullet \begin{pmatrix}1\\1\\-1\end{pmatrix} = 4-1-3 = 0$ $\quad \begin{pmatrix}0\\1\\1\end{pmatrix} \bullet \begin{pmatrix}1\\1\\-1\end{pmatrix} = 1-1 = 0$

$\begin{pmatrix}1\\-1\\0\end{pmatrix} \bullet \begin{pmatrix}1\\1\\-1\end{pmatrix} = 1-1 = 0$ $\quad \begin{pmatrix}3\\1\\4\end{pmatrix} \bullet \begin{pmatrix}1\\1\\-1\end{pmatrix} = 3+1-4 = 0$

Showing that **n** is perpendicular to all the defining directions of the planes and hence that the planes are parallel.

An alternative method would be for each plane, to use the cross product to find a vector perpendicular to that plane and then show that these two perpendicular vectors were multiples of **n**.

b. $\mathbf{t} = \begin{pmatrix}2\\-1\\0\end{pmatrix} + \delta\begin{pmatrix}1\\1\\-1\end{pmatrix}$

c. Solving $\begin{cases}2+\delta = 0+\alpha+3\beta\\-1+\delta = -5-\alpha+\beta\\0-\delta = 3+4\beta\end{cases}$ by calculator gives $\delta = -3, \alpha = -1, \beta = 0$

So, intersection point is $(-1, -4, 3)$

d. Require distance between $(2, -1, 0)$ and $(-1, -4, 3)$

Which is $\sqrt{3^2+3^2+3^2} = 3\sqrt{3}$

33. **a.** Period $= 12 = \dfrac{2\pi}{b} \Rightarrow b = \dfrac{\pi}{6}$

a is amplitude $= \dfrac{11-1}{2} \Rightarrow a = 5$

c is vertical translation so $c = 6$

$h = 5\sin\left(\dfrac{\pi}{6}(t-3)\right) + 6$

b. **i.**

t	0	1	2	3	4	5	6
h	1	1.670	3.5	6	8.5	10.330	11

Changes in each hour are:

0.670, 1.83, 2.5, 2.5, 1.83, 0.670 (3 sf)

ii. To convert to the required ratio, we need to multiply by $\dfrac{12}{10}$, giving

0.80: 2.2: 3: 3: 2.2: 0.80 (2 sf)

iii. This shows that the model has reasonably good agreement with Sinbad's rule.

c. i. $\dfrac{dh}{dt} = 5 \times \dfrac{\pi}{6} \cos\left(\dfrac{\pi}{6}(t-3)\right)$

ii. The maximum rate of change is $\dfrac{5\pi}{6} = 2.62$ m per hour (3 sf)

This occurs when $t = 3$ at 3:00 a.m.

HL Paper 3: Technology required

34. a. In the sine rule $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ we only know one of each pair and so cannot proceed any further.

b. $c^2 = 5^2 + 3^2 - 2 \times 5 \times 3 \cos 80° \Rightarrow c = 5.36568... = 5.37$ (3 sf)

c. $A + B = 180 - 80 = 100 \Rightarrow \dfrac{A+B}{2} = 50$

$$\frac{5-3}{5+3} = \frac{\tan\left(\dfrac{A-B}{2}\right)}{\tan 50°} \Rightarrow \tan\left(\frac{A-B}{2}\right) = \frac{\tan 50°}{4} \Rightarrow \frac{A-B}{2} = 16.5908...$$

$A = 66.5908... = 66.6°$ (3 sf)

$B = 33.4091... = 33.4°$ (3 sf)

d. $\dfrac{5}{\sin 66.5908...°} = \dfrac{c}{\sin 80°} \Rightarrow c = 5.36568... = 5.37$(3 sf)

This agrees with the cosine rule answer.

e. Let $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$

$\dfrac{a-b}{a+b} = \dfrac{k\sin A - k\sin B}{k\sin A + k\sin B} = \dfrac{\sin A - \sin B}{\sin A + \sin B}$, as required

f. i. $\sin(C+D) + \sin(C-D)$

$= \sin C \cos D + \cos C \sin D + \sin C \cos(-D) + \cos C \sin(-D)$

$= \sin C \cos D + \cos C \sin D + \sin C \cos D - \cos C \sin D$

Then, using odd and even properties of sin and cos:

$= 2\sin C \cos D$, as required

ii. $\sin(C+D) - \sin(C-D)$

$= \sin C \cos D + \cos C \sin D - (\sin C \cos(-D) + \cos C \sin(-D))$

$= \sin C \cos D + \cos C \sin D - \sin C \cos D + \cos C \sin D$

Using odd and even properties of sin and cos:

$= 2\cos C \sin D$, as required

g. Need $C + D = A, C - D = B \Rightarrow C = \dfrac{A+B}{2}, D = \dfrac{A-B}{2}$

$\sin A + \sin B = 2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

$\sin A - \sin B = 2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$

h. $$\frac{\sin A - \sin B}{\sin A + \sin B} = \frac{2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}}{2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}} = \frac{\sin\dfrac{A-B}{2}}{\cos\dfrac{A-B}{2}} \Bigg/ \frac{\sin\dfrac{A+B}{2}}{\cos\dfrac{A+B}{2}} = \frac{\tan\dfrac{A-B}{2}}{\tan\dfrac{A+B}{2}}$$

Completing the proof

4 STATISTICS AND PROBABILITY: END-OF-CHAPTER PRACTICE QUESTION WORKED SOLUTIONS

SL Paper 1: Section A
No technology allowed

1. The median is the middle value, so $r = 5$

The mode must occur more than any other value so there must be two 7's

hence $s = t = 7$

The range is maximum – minimum $= 7 - p = 5 \Rightarrow p = 2$

Mean $= \dfrac{2+q+5+7+7}{5} = 5 \Rightarrow q = 4$

2. **a.** **i.** 1 **ii.** 0 **iii.** 1

iv. $P(A\,|\,B)+P(A'\,|\,B)=\dfrac{P(A\cap B)}{P(B)}+\dfrac{P(A'\cap B)}{P(B)}=\dfrac{P(B)}{P(B)}=1$

b. $P(A\cap B\,|\,B)=\dfrac{P(A\cap B\cap B)}{P(B)}=\dfrac{P(A\cap B)}{P(B)}=P(A\,|\,B)$

3. **a.** Placing data in order:

Set A: 1,2,**25**,32,33,34,35,36,**37**,60,65

$Q_1 = 25$ $Q_3 = 37$ IQR = 37 – 25 = 12

$Q_3 + 1.5 \times \text{IQR} = 37 + 18 = 55$ $Q_1 - 1.5 \times \text{IQR} = 25 - 18 = 7$

So, 1, 2, 60 and 65 are outliers

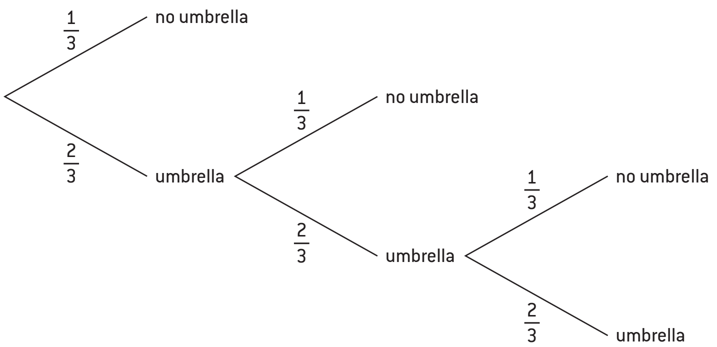
b. New data in order Set B: 25,**32**,33,34,35,**36**,37

$Q_1 = 32$ $Q_3 = 36$ IQR = 36 – 32 = 4

$Q_3 + 1.5 \times \text{IQR} = 36 + 6 = 42$ $Q_1 - 1.5 \times \text{IQR} = 32 - 6 = 26$

So, 25 is an outlier for this set.

4. **a.**



b. Probability of lost umbrella is $1-\left(\dfrac{2}{3}\right)^3=\dfrac{19}{27}$ $\left(\text{or } \dfrac{1}{3}+\dfrac{2}{3}\times\dfrac{1}{3}+\dfrac{2}{3}\times\dfrac{2}{3}\times\dfrac{1}{3}=\dfrac{19}{27}\right)$

c. $P(\text{2nd}\,|\,\text{lost})=\dfrac{P(\text{lost in 2nd})}{P(\text{lost})}=\dfrac{\frac{2}{3}\times\frac{1}{3}}{\frac{19}{27}}=\dfrac{6}{19}$

5. **a.** Expected values: A: $\frac{8}{3}$ B: 3 C: $\frac{10}{3}$ D: 3

b. **i.** $\frac{4}{6}=\frac{2}{3}$ **ii.** $\frac{4}{6}=\frac{2}{3}$ **iii.** $\frac{2}{6}+\frac{4}{6}\times\frac{1}{2}=\frac{2}{3}$ **iv.** $\frac{1}{2}+\frac{1}{2}\times\frac{2}{6}=\frac{2}{3}$

c. Pretend to be magnanimous and let them choose first. Then choose the dice that beats theirs on average.

6. **a.** Independent implies $P(A \cap B) = P(A) \times P(B)$. Proof by contradiction.

Suppose A and B are mutually exclusive. Then:

$P(A \cap B) = 0 \Rightarrow P(A) \times P(B) = 0 \Rightarrow P(A) = 0$ or $P(B) = 0$

This gives the desired contradiction, so A and B cannot be mutually exclusive.

b. **i.** $P(A) = 0 \Rightarrow P(A \cap B) = 0$ so A and B are mutually exclusive.

ii. $P(A \cap B) = 0$ and $P(A) \times P(B) = 0$ since $P(A) = 0$

so A and B are independent.

7. **a.** Let the coins be labeled $H_1 | H_2$, $T_1 | T_2$, $H_3 | T_3$. Of the six original possibilities, only three can now have occurred and they are all equally likely:

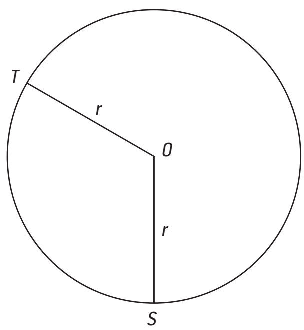
$$\frac{H_1}{H_2} \qquad \frac{H_2}{H_1} \qquad \frac{H_3}{T_3}$$

So, Kate's claim is wrong, and the probability that the other side of the coin is also a head must is $\frac{1}{3}$

b. **i.** The youngest child is either a boy or a girl so the probability is $\frac{1}{2}$

ii. There are 3 possibilities: BB, BG or GB, so the probability is $\frac{1}{3}$

8.



Using the diagram above and the cosine rule:

$$3r^2 = r^2 + r^2 - 2r^2 \cos S\hat{O}T \Rightarrow \cos S\hat{O}T = \frac{-1}{2} \Rightarrow S\hat{O}T = 120°$$

So, the train will be more than $\sqrt{3}r$ from the station if it is on a top arc with central angle of $360 - 120 - 120 = 120°$. The probability is $\frac{120}{360} = \frac{1}{3}$

9. a. $\frac{1}{7} \times \frac{1}{7} = \frac{1}{49}$

b. 7 possible days so $\frac{7}{49} = \frac{1}{7}$

c. There are 6 possible consecutive days: Mon/Tue, Tue/Wed, Wed/Thur, Thur/Fri, Fri/Sat, Sat/Sun. With each of these the children could be either way round so the probability is

$\frac{2 \times 6}{49} = \frac{12}{49}$

10. a. ${}^6C_3 = 20$

b. i. $\frac{1}{2}$, the same for all players

ii. Not a random sample, not all samples have the same probability of being chosen.

c. i. 1 female and 2 males

ii. ${}^2C_1 \times {}^4C_2 = 2 \times 6 = 12$

d. Convenience sampling

11. $A \quad F_2 \quad BW_1$

$B \quad F_3 \quad BW_3$

$C \quad F_1 \quad BW_2$

SL Paper 1: Section B
No technology allowed

12. a. $P((A \cup B) \cup C) = P(A \cup B) + P(C) - P((A \cup B) \cap C)$

$= P(A) + P(B) - P(A \cap B) + P(C) - P((A \cap C) \cup (B \cap C))$

$= P(A) + P(B) - P(C) - P(A \cap B) - (P(A \cap C) + P(B \cap C) - P(A \cap C \cap (B \cap C)))$

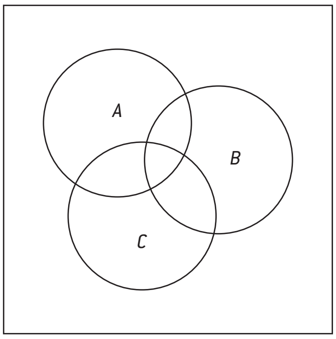
$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

as required.

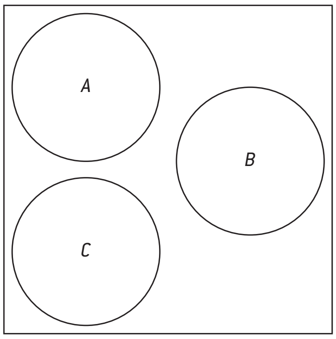
b. $P(A \cup B \cup C) = P(A) + P(B) + P(C)$

c. $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap C)$

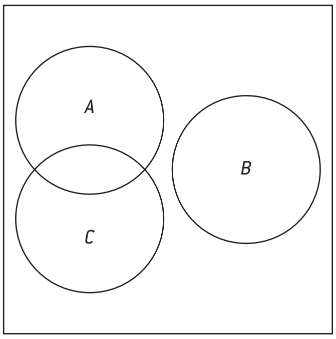
d. $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C)$



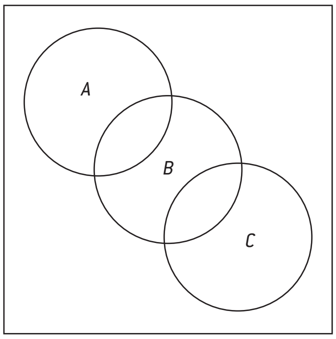
i.



ii.



iii.



iv.

f. Since there are no even primes except 2, events A and C are mutually exclusive.

i. $P(A \cup B \cup C) = \frac{30}{300} + \frac{100}{300} + \frac{150}{300} - \frac{2}{300} - \frac{50}{300} = \frac{228}{300}\left(= \frac{19}{25}\right)$

ii. Using the diagram in **e. iv.**: $P((B \cap A') \cap C') = \frac{100}{300} - \frac{2}{300} - \frac{50}{300}$

$= \frac{48}{300}\left(= \frac{4}{25}\right)$

13. a. $P(A) = \frac{1}{4}, P(B) = \frac{1}{5}$ $\quad$ $A \cap B$ means being divisible by 20

$P(A \cap B) = \frac{1}{20}$ $\quad$ $\frac{1}{20} = \frac{1}{4} \times \frac{1}{5}$ so A and B are independent.

b. $P(A) = \frac{1}{4}, P(C) = \frac{1}{10}$ $\quad$ $A \cap C$ means being divisible by 20

$P(A \cap C) = \frac{1}{20}$ $\quad$ $\frac{1}{20} \neq \frac{1}{4} \times \frac{1}{10}$ so A and C are not independent.

c. It is the fact that 4 and 10 have a common prime factor, whereas 4 and 5 do not, which makes A and C not independent whereas A and B are independent.

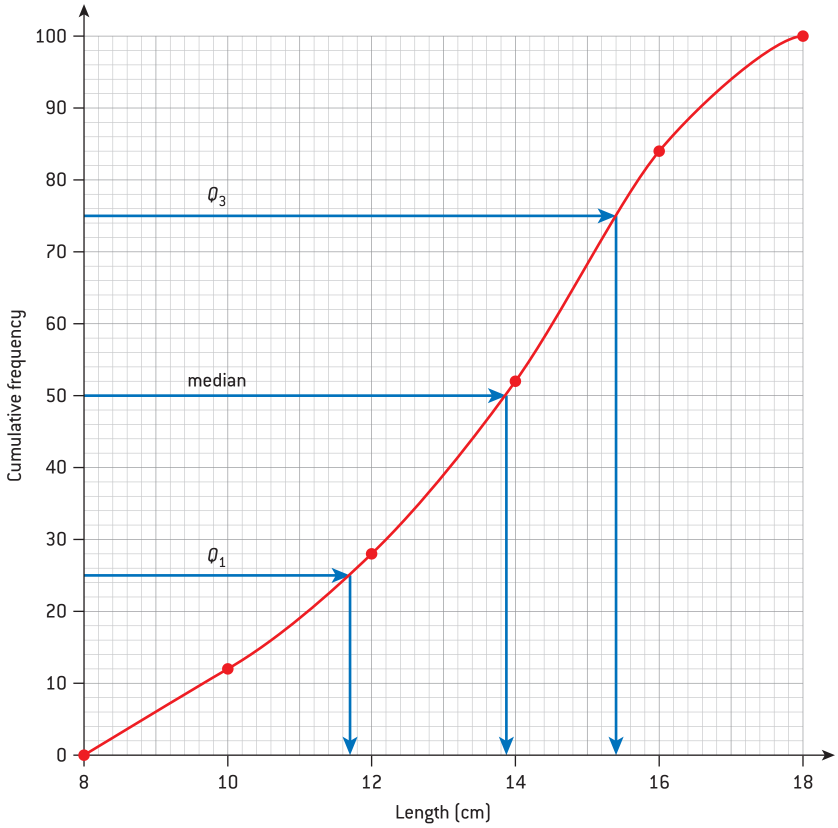
14. a. 12 + 16 + 24 + 32 + 16 = 100

100 leaves were measured.

b.

Length (cm)	Cumulative frequency
$l \leq 8$	0
$l \leq 10$	12
$l \leq 12$	28
$l \leq 14$	52
$l \leq 16$	84
$l \leq 18$	100

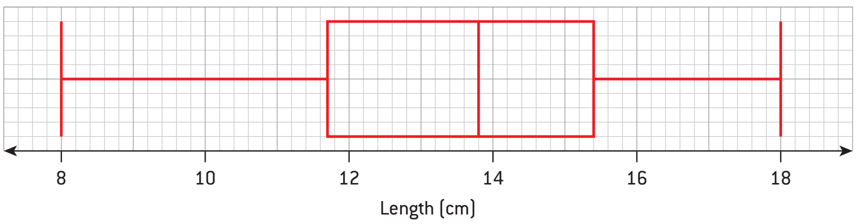
c.



d. **i.** Median = 13.8 cm

ii. IQR = $Q_3 - Q_1$ = 15.4 – 11.7 = 3.7 cm

e.



SL Paper 2: Section A
Technology required

15. Let X be the number of successful kicks. $X \sim B(10, 0.9)$

a. $E(X) = 10 \times 0.9 = 9$

b. $P(X = 8) = 0.194$ (3 sf)

c. $P(X \leq 5) = 0.00163$ (3 sf)

d. $\dfrac{1}{10}$

16. $D \sim \mathrm{N}\,(2.60, 0.08^2)$

a. $P(D < 2.56) = 0.3085\ldots$ 30.9% (3 sf)

b. $P(D > u) = 0.05 \Rightarrow u = 2.7315\ldots$

$P(D < l) = 0.04 \Rightarrow l = 2.4599\ldots$

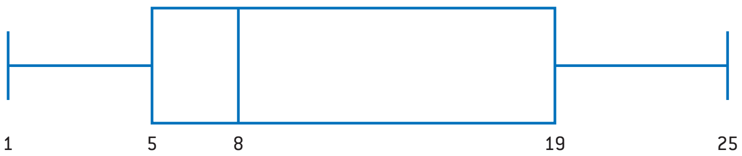
So, acceptable limits are $2.46 < D < 2.73$ (3 sf)

c. $P(D < 2.70 | D > 2.60) = \dfrac{P(2.60 < D < 2.70)}{0.5} = 0.789$ (3 sf)

17. **a.** **i.** 7 **ii.** 8 **iii.** 10.8

b. **i.** $25 - 1 = 24$ **ii.** $19 - 5 = 14$ **iii.** 7.13 (3 sf)

c.



18. **a.**

Height, h m	Frequency
$1.05 < h \le 1.10$	1
$1.10 < h \le 1.15$	20
$1.15 < h \le 1.20$	30
$1.20 < h \le 1.25$	25
$1.25 < h \le 1.30$	22
$1.30 < h \le 1.35$	2

Height, h m	Cumulative frequency
$h \le 1.10$	1
$h \le 1.15$	21
$h \le 1.20$	51
$h \le 1.25$	76
$h \le 1.30$	98
$h \le 1.35$	100

b. Modal class is $1.15 < h \le 1.20$

c. Using mid-points of class widths:

i. $\bar{h} \simeq 1.2$ m

ii. $\sigma \simeq 0.0854$ m (3 sf)

SL Paper 2: Section B
Technology required

19. a. 50%

b. $P(-l < X < l) = 0.5 \Rightarrow l = 0.674489\ldots = 0.674$ (3 sf)

c. Outlier if:

$y > Q_3 + 1.5 \times IQR = Q_3 + 1.5 \times (Q_3 - Q_1) = 2.5Q_3 - 1.5Q_1$

or

$y < Q_1 + 1.5 \times IQR = Q_1 + 1.5 \times (Q_3 - Q_1) = 2.5Q_1 - 1.5Q_3$

as required

d. By symmetry, $P(X > 4l \text{ or } X < -4l) = 2P(X > 4l) = 0.00698$ (3 sf)

e. Similarity is between Q_1 and $-l$ and also Q_3 and l.

So $2.5Q_3 - 1.5Q_1$ is similar to $2.5l - 1.5(-l) = 4l$

f. If $W \sim \text{N}(\mu, \sigma^2)$ then $X = \dfrac{W - \mu}{\sigma} \sim \text{N}(0, 1^2)$

$P(W > \mu + 4l\sigma \text{ or } X < \mu - 4l\sigma) = P(X > 4l \text{ or } X < -4l)$

The answer is also 0.00698 (3 sf)

20. a. $X = 1$ comes from sequence 6, probability is $\dfrac{1}{6}$

$X = 2$ comes from sequence $\cancel{6}6$, probability is $\dfrac{5}{6} \times \dfrac{1}{6} = \dfrac{5}{6^2}$

$X = 3$ comes from sequence $\cancel{6}\cancel{6}6$, probability is $\dfrac{5}{6} \times \dfrac{5}{6} \times \dfrac{1}{6} = \dfrac{25}{6^3}$

$X = 4$ comes from sequence $\cancel{6}\cancel{6}\cancel{6}$ anything, probability is $\dfrac{5}{6} \times \dfrac{5}{6} \times \dfrac{5}{6} = \dfrac{125}{6^3}$

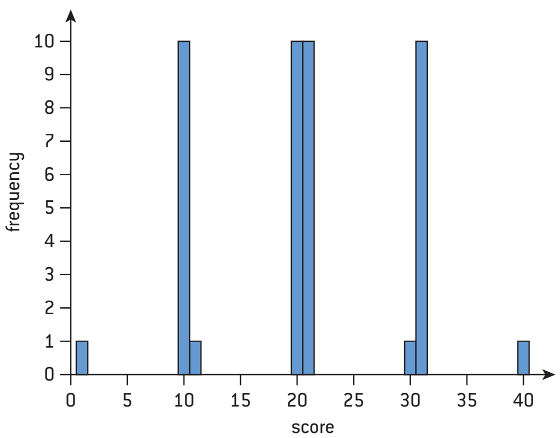
x	1	2	3	4
$P(X = x)$	$\dfrac{1}{6}$	$\dfrac{5}{36}$	$\dfrac{25}{216}$	$\dfrac{125}{216}$

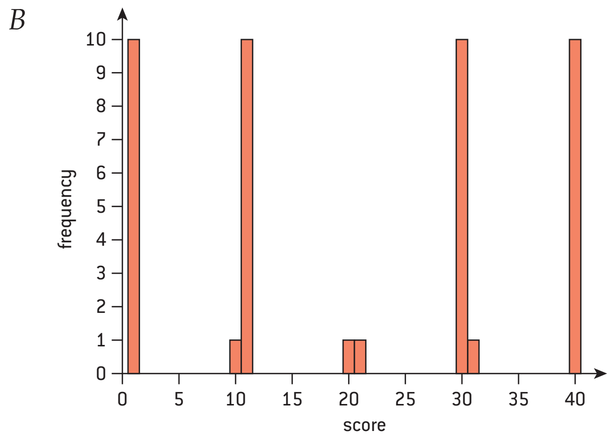
b. Sum of probabilities should be 1. Checking: $\dfrac{36 + 30 + 25 + 125}{6^3} = \dfrac{216}{6^3} = 1$

c. i. Mode is 4 **ii.** By calculator, $E(X) = \dfrac{671}{216} = 3.11$ (3 sf)

iii. By calculator, $\text{Var}(X) = 1.37$ (3 sf)

21. a. A





b. A:

i. 20.5 **ii.** 20.5 **iii.** $40 - 1 = 39$ **iv.** $30.5 - 10.5 = 20$
v. 8.46 (3 sf)

B:

i. 20.5 **ii.** 20.5 **iii.** $40 - 1 = 39$ **iv.** $30.5 - 10.5 = 20$
v. 14.8 (3 sf)

c. Visually, the histogram of set B is more dispersed (spread-out) than that of set A.

The ranges and the interquartile ranges are the same. It is only the standard deviation that make a distinction, with that of set B being greater. So, the standard deviation was the best indicator of dispersion.

22. $L \sim \mathrm{N}\,(20, 0.5)^2$

a. $P(19.6 < L < 20.8) = 0.733$ (3 sf)

b. $P(L > l) = 0.95 \Rightarrow l = 19.2$ cm (3 sf)

c. $L \sim \mathrm{N}\,(\mu, 0.5^2)$ $\quad Z = \dfrac{L-\mu}{0.5}$ $\quad Z \sim \mathrm{N}\,(0, 1^2)$

$P(L > 19) = 0.8 \Rightarrow P\left(Z < \dfrac{19-\mu}{0.5}\right) = 2 \qquad \dfrac{19-\mu}{0.5} = -0.84162\ldots$

New $\mu = 19.4208\ldots = 19.4$ cm (3 sf)

d. $L \sim \mathrm{N}\,(19.4208\ldots, 0.5^2)$ $\quad P(L < 20) = 0.877 = 87.7\%$ (3 sf)

23. a. $r = -0.840$ (3 sf)

b. Fairly strong, negative, linear correlation

c. Require the regression line y on x

$y = -0.1440\ldots x + 16.58\ldots$

Substituting $x = 105$ gives $y = 1.46$, so the estimate is 1, to the nearest whole number

d. Require the regression line x on y

$x = -4.894\ldots y + 110.03\ldots$

Substituting $y = 5$ gives estimate as 85.6 (3 sf), which is 86 to the nearest integer.

HL Paper 1: Section A
No technology allowed

24. **a.** A and B being independent implies $P(A \cap B) = P(A) \times P(B)$

$P(A') \times P(B) = (1 - P(A))\,P(B) = P(B) - P(A)\,P(B) = P(B) - P(A \cap B)$
$= P(A' \cap B)$

Showing that A' and B are also two independent events.

b. From part **a.**, A and B independent implies A' and B independent which implies B and A' independent.

Part **a.** shows X and Y independent implies X' and Y independent.

Substituting in respectively, B and A' independent implies B' and A' independent which in turn implies A' and B' independent.

So A and B independent implies A' and B' independent

An alternative proof is:

$P(A') \times P(B') = (1 - P(A))(1 - P(B)) = 1 - P(A) - P(B) + P(A)P(B)$

$= 1 - P(A) - P(B) + P(A \cap B) = 1 - P(A \cup B) = P((A \cup \mathrm{B})')$

Which upon considering a Venn diagram $= P(A' \cap B')$ as required.

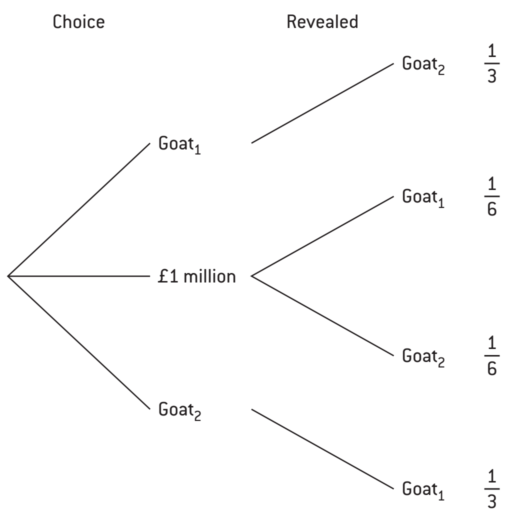
25. **a.** There are 3! = 6 ways of putting the three S's next to each other. Think of this as being one entity. We have 5 objects to be placed in a line which can be done in 5! ways. The original seven letters could be placed in 7! ways. The answer is:

$$\frac{6 \times 5!}{7!} = \frac{1}{7}$$

b. There are 2 ways of putting the C's at the ends. The remaining 5 letters can be placed in the middle in 5! ways. The answer is:

$$\frac{2 \times 5!}{7!} = \frac{1}{21}$$

26.



Note that the game's host would never reveal the million pounds, as that would ruin the game.

In the top and bottom branches, which have a combined probability of $\frac{2}{3}$, it would be better to make the change. In the middle two branches which have a combined probability of $\frac{1}{3}$ it would not be a good idea make the change.

So, advise the contestant to change their choice.

27. **a.** $r = -0.148$ (3 sf)

b. Extremely weak, negative, linear correlation

c. As established in part **a.**, there is essentially no correlation, so the line of best fit is meaningless.

This would be extrapolation it is well away from the data set.

It is the height that is known, so should be using the x on y regression line (if any).

HL Paper 1: Section B
No technology allowed

28. **a.** $\int_a^b (x-\mu)^2 f(x)\mathrm{d}x = \int_a^b (x^2 - 2x\mu + \mu^2)^2 f(x)\mathrm{d}x$

$\int_a^b x^2 f(x)\mathrm{d}x - 2\mu\int_a^b xf(x)\mathrm{d}x + \mu^2\int_a^b f(x)\mathrm{d}x = \int_a^b x^2 f(x)\mathrm{d}x - 2\mu(\mu) + \mu^2$

(since the area under the curve must equal 1)

$= \int_a^b x^2 f(x)\mathrm{d}x - \mu^2$, as required.

b. $\mathrm{Var}(X) = E(X^2) - (E(X))^2$

$\mathrm{Var}(X) \geq 0 \Rightarrow E(X^2) \geq (E(X))^2$, as required.

c. Equality implies $\mathrm{Var}(X) = \int_a^b (x-\mu)^2 f(x)\mathrm{d}x = 0$. Now since $(x-\mu)^2 \geq 0$ and $f(x) \geq 0$ we must have $(x-\mu)^2 f(x) = 0$ for all $x \in [a, b]$. So when $x \neq \mu$ then $f(x) = 0$ which makes it impossible to have $\int_a^b f(x)\mathrm{d}x = 1$.

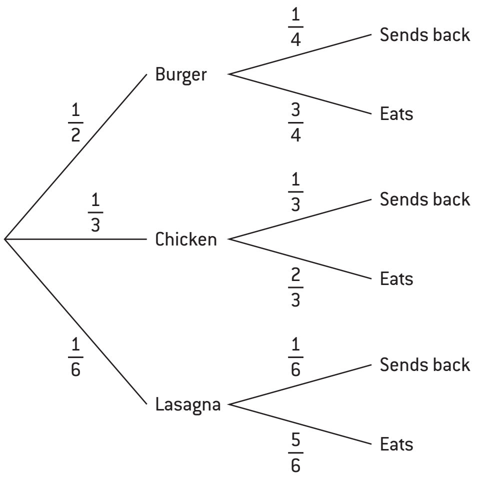
By contradiction, we have shown that equality is not possible for a continuous random variable.

d. **i.** $E(Y) = 7 \times 1 = 7$ **ii.** $E(Y^2) = 7^2 \times 1 = 49$ **iii.** $\mathrm{Var}(Y) = 0$

So $E(Y^2) = (E(Y))^2$ is possible for a discrete random variable.

HL Paper 2: Section A
Technology required

29. a.



b. $P(\text{sends back}) = \frac{1}{2} \times \frac{1}{4} + \frac{1}{3} \times \frac{1}{3} + \frac{1}{6} \times \frac{1}{6} = \frac{19}{72}$

c. $P(C \mid S) = \dfrac{P(C \cap S)}{P(S)} = \dfrac{\frac{1}{9}}{\frac{19}{72}} = \dfrac{8}{19}$

d. $P(B \mid S') = \dfrac{P(B \cap S')}{P(S')} = \dfrac{\frac{1}{2} \times \frac{3}{4}}{1 - \frac{19}{72}} = \dfrac{\frac{3}{8}}{\frac{53}{72}} = \dfrac{27}{53}$

30. i. $\dfrac{{}^{6}C_3}{{}^{15}C_3} = \dfrac{4}{91}$ **ii.** $\dfrac{{}^{5}C_2 \times 4}{{}^{15}C_3} = \dfrac{8}{91}$ **iii.** $\dfrac{4 \times 5 \times 6}{{}^{15}C_3} = \dfrac{24}{91}$

iv. 3B, 3R or 3G $\dfrac{{}^{6}C_3 + {}^{5}C_3 + {}^{4}C_3}{{}^{15}C_3} = \dfrac{34}{455}$

HL Paper 2: Section B
Technology required

31. a. Probabilities must add to 1, so $\dfrac{4+3+2}{13} + k = 1 \Rightarrow k = \dfrac{4}{13}$, as required

b. i. By calculator, $E(X) = \dfrac{32}{13} = 2.46$ (3 sf)

ii. By calculator, $\text{Var}(X) = (0.1216\ldots)^2 = 1.48$ (3 sf)

c.

y	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
$P(Y = y)$	$\frac{4}{13}$	$\frac{3}{13}$	$\frac{2}{13}$	$\frac{4}{13}$

By calculator, $E(Y) = \dfrac{43}{78} = 0.551$ (3 sf)

d. $\dfrac{43}{78} \neq \dfrac{13}{22}$

e. $E(W) = 0$ since the game is fair

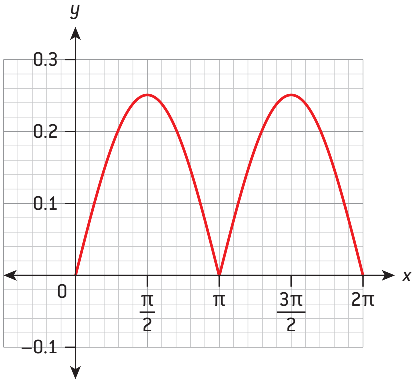
f. $E\left(\dfrac{1}{W}\right)$ will exist but $\dfrac{1}{0}$ is undefined.

32. a. $\int_0^{2\pi} f(x)\,dx = 1 \Rightarrow \int_0^{\pi} k\sin x\,dx - \int_0^{2\pi} k\sin x\,dx = 1$

$[-k\cos x]_0^{\pi} + [k\cos x]_{\pi}^{2\pi} = 1 \Rightarrow (k+k)+(k+k)=1 \Rightarrow 4k=1$

$\Rightarrow k = \frac{1}{4}$, as required

b.



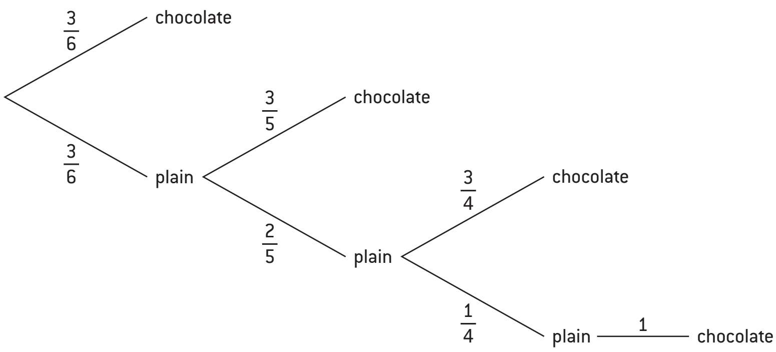
c. i. $E(X) = \pi$, by symmetry

ii. Median $= \pi$, by symmetry

iii. There are maximums of height $\frac{1}{4}$ at both $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$

d. $\mathrm{Var}(X) = \int_0^{2\pi} \frac{x^2}{4}|\sin x|\,dx - \pi^2 = 2.93$ (3 sf)

33. a.



Let C = chocolate, and P = plain in the following.

b. $X = 0$ comes from sequence C, probability is $\frac{1}{2}$

$X = 1$ comes from sequence PC, probability is $\frac{1}{2} \times \frac{3}{5} = \frac{3}{10}$

$X = 2$ comes from sequence PPC, probability is $\frac{1}{2} \times \frac{2}{5} \times \frac{3}{4} = \frac{3}{20}$

$X = 3$ comes from sequence PPPC, probability is $\frac{1}{2} \times \frac{2}{5} \times \frac{1}{4} \times 1 = \frac{1}{20}$

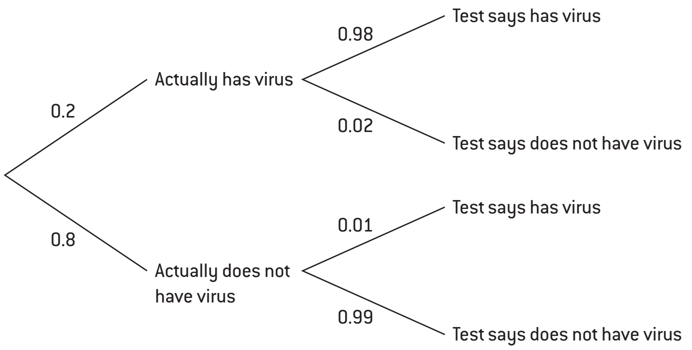
x	0	1	2	3
$P(X = x)$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{3}{20}$	$\frac{1}{20}$

Sum of probabilities should be 1. Checking $\frac{10}{20} + \frac{6}{20} + \frac{3}{20} + \frac{1}{20} = \frac{20}{20} = 1$

c. By calculator $E(X) = \frac{3}{4}$

d. By calculator $\mathrm{Var}(X) = \frac{63}{80}$

34. **a.**



b. Probability test says has virus = 0.2 × 0.98 + 0.8 × 0.01 = 0.204

c. $P(V \mid \text{test says yes}) = \dfrac{P(V \cap \text{test says yes})}{P(\text{test says yes})} = \dfrac{0.2 \times 0.98}{0.204} = 0.961\ (3\text{ sf})$

d. $P(V \mid \text{test says no}) = \dfrac{P(V \cap \text{test says no})}{P(\text{test says no})} = \dfrac{0.2 \times 0.02}{1 - 0.204} = 0.00503\ (3\text{ sf})$

e. Let X be the number of pupils that the tests says have the virus.
$X \sim B(300, 0.01)$

Expected value $E(X) = np \quad = 300 \times 0.01 = 3$ pupils

f. $P(X \geq 1) = 1 - P(X = 0) = 0.951$ (3 sf)

35. **a.** **i.** $\dfrac{6}{8} \times \dfrac{5}{7} \times \dfrac{4}{6} = \dfrac{5}{14}$ **ii.** 0

b.

x	0	1	2	3
$P(X = x)$	0	$\dfrac{{}^6C_1 \times {}^2C_2}{{}^8C_3} = \dfrac{3}{28}$	$\dfrac{{}^6C_2 \times {}^2C_1}{{}^8C_3} = \dfrac{15}{28}$	$\dfrac{{}^6C_3}{{}^8C_3} = \dfrac{5}{14}$

$E(X) = \dfrac{3 \times 1 + 15 \times 2 + 10 \times 3}{28} = \dfrac{63}{28} = \dfrac{9}{4}$

c. **i.** $\left(\dfrac{6}{8}\right)^3 = \dfrac{27}{64}$ **ii.** $\left(\dfrac{2}{8}\right)^3 = \dfrac{1}{64}$

d. This situation is binomial $E(Y) = 3 \times \dfrac{6}{8} = \dfrac{9}{4}$

e. The expected value is the same regardless as to whether there is replacement or not.

HL Paper 3: Technology required

36. **a.** **i.** $E(X) = \dfrac{1+2+3+4+5+6}{6} = \dfrac{7}{2}$

ii. $\text{Var}(X) = E(X^2) - (E(X))^2 = \dfrac{1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2}{6} - \dfrac{49}{4} = \dfrac{35}{12}$

b.

$Y \backslash X$	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Counting $P(T=6)=\frac{6}{36}=\frac{1}{6}$

c.

t	2	3	4	5	6	7	8	9	10	11	12
$P(T=t)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Mode is 7

d. Using a calculator: **i.** $E(T)=7$, **ii.** $\text{Var}(T)=\frac{35}{6}$

e. $E(T)=7$, $E(X)+E(Y)=\frac{7}{2}+\frac{7}{2}=7$ confirming the result.

f. X and Y are independent

$\text{Var}(T)=\frac{35}{6}$, $\text{Var}(X)+\text{Var}(Y)=\frac{35}{12}+\frac{35}{12}=\frac{35}{6}$ confirming the result.

g. $\overline{T}\approx N\left(7,\frac{35}{600}\right)$, $P(6.8\le\overline{T}\le 7.2)=0.592$ (3 sf)

h. Considering another lattice diagram

$Y \backslash X$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	2	2	2	2
3	1	2	3	3	3	3
4	1	2	3	4	4	4
5	1	2	3	4	5	5
6	1	2	3	4	5	6

$P(M=2)=\frac{9}{36}=\frac{1}{4}$

i. $2M=T\Rightarrow 2X=X+\text{Y}$ or $2Y=X+Y\Rightarrow X=Y$

So, the probability is $\frac{6}{36}=\frac{1}{6}$

5 CALCULUS: END-OF-CHAPTER PRACTICE QUESTION WORKED SOLUTIONS

SL Paper 1: Section A
No technology allowed

1. **a.** $\frac{dy}{dx} = 6x + 2$

b. $y = x^3 + x^{-3} \Rightarrow \frac{dy}{dx} = 3x^2 - 3x^{-4}$

c. $y = x^{\frac{1}{3}} \Rightarrow \frac{dy}{dx} = \frac{1}{3}x^{\frac{-2}{3}}$

2. **a.** $\frac{dy}{dx} = 4x^3$

At $x = 1$, $\frac{dy}{dx} = 4$ The equation of tangent is $y = 4x + c$

Through $(1, 2) \Rightarrow 2 = 4 + c \Rightarrow c = -2$, equation of tangent is $y = 4x - 2$

An equally good alternative is to give the answer in the form $y - 2 = 4(x - 1)$

b. At $x = -1$, $\frac{dy}{dx} = -4$, gradient of normal is $\frac{1}{4}$, equation of normal is $y = \frac{1}{4}x + d$

Through $(-1, 2) \Rightarrow 2 = -\frac{1}{4} + d \Rightarrow d = \frac{9}{4}$, equation of normal is $y = \frac{1}{4}x + \frac{9}{4}$

An equally good alternative is to give the answer in the form $y - 2 = \frac{1}{4}(x + 1)$

3. **a.** $\frac{dy}{dx} = x^2 - 3x - 10 = (x + 2)(x - 5)$ $\frac{dy}{dx} = 0 \Rightarrow x = -2 \text{ or } 5$

x	$x < -2$	-2	$-2 < x < 5$	5	$x > 5$
$\frac{dy}{dx}$	+ve	0	–ve	0	+ve

Maximum when $x = -2$ Minimum when $x = 5$

b. **i.** increasing for $]-\infty, -2[\cup]5, +\infty[$ **ii.** decreasing for $]-2, 5[$

4. **a.** $\frac{dy}{dx} = 3(x - 1)^2$, $\frac{d^2y}{dx^2} = 6(x - 1) = 0 \Rightarrow x = 1$, so the point of inflexion is (1, 5)

b. Using a sign table

x	$x < 1$	1	$x > 1$
$\frac{d^2y}{dx^2}$	–ve	0	+ve

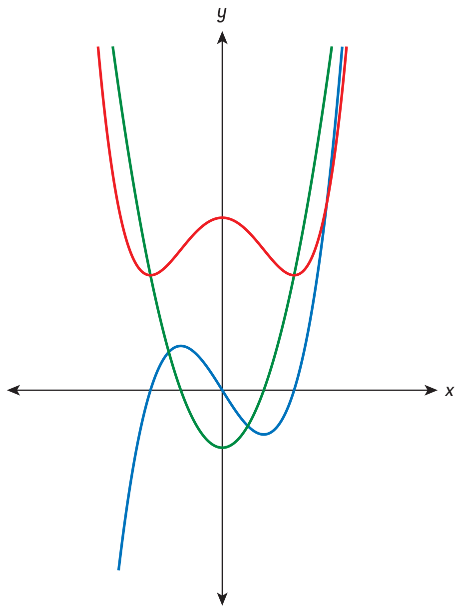
There is a sign change at $x = 1$, hence there is a point at $x = 1$.

$\frac{dy}{dx} = 0$ at $x = 1$ so the tangent is a horizontal one.

c. i. concave up for $\{x \in \mathbb{R} | x > 1\}$

ii. concave down for $\{x \in \mathbb{R} | x < 1\}$

5.



$f'(x)$ is in blue and passes through the origin.

$f''(x)$ is in green.

These graphs were obtained using the following information.

$f'(x)$ will have zeros at turning points of $f(x)$, it will be positive when $f(x)$ is increasing and negative when $f(x)$ is decreasing.

$f''(x)$ will have zeros at the inflection points of $f(x)$ (which will correspond to the turning points of $f'(x)$), it will be positive when $f(x)$ is concave up ($f'(x)$ is increasing) and negative when $f(x)$ is concave down ($f'(x)$ is decreasing).

6. a. Integrating $f'(x)$ gives $f(x) = x^3 + e^{x-1} + x + c$

$f(1) = 2 \Rightarrow 1 + 1 + 1 + c = 2 \Rightarrow c = -1$

$f(x) = x^3 + e^{x-1} + x - 1$

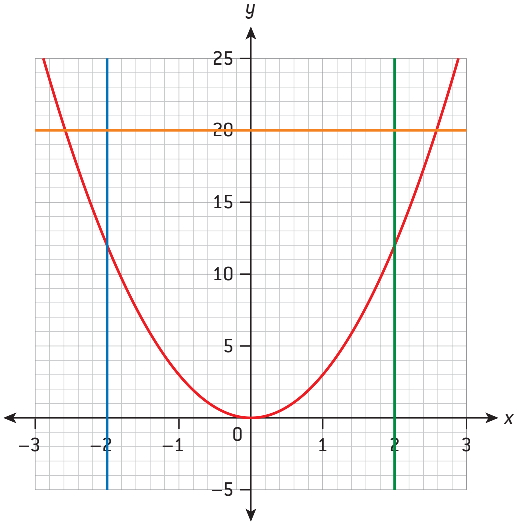
b. Differentiating $f'(x)$ gives $f''(x) = 6x + e^{x-1}$

7. a. $\int x^2 + x^{-2}\,dx = \frac{x^3}{3} - x^{-1} + c$

b. $\int x^{\frac{1}{2}}\,dx = \frac{2}{3}x^{\frac{3}{2}} + c$

c. $\int \sin x + \cos 3x\,dx = -\cos x + \frac{1}{3}\sin 3x + c$

8. **a.**



b. Require $\int_{-2}^{2} 20-3x^2\mathrm{d}x=\left[20x-x^3\right]_{-2}^{2}=(32)-(-32)=64$

9. **a.** By inspection, $\int \sin x(\cos x)^5\,\mathrm{d}x=\frac{-1}{6}(\cos x)^6+c$

b. By inspection, $\int xe^{x^2+3}\,\mathrm{d}x=\frac{1}{2}e^{x^2+3}+c$

c. $\int \frac{x^2}{(x^3+2)^2}\mathrm{d}x=\int x^2(x^3+2)^{-2}\,\mathrm{d}x=\frac{-1}{3}(x^3+2)^{-1}+c$ by inspection.

Note all the above could also be done by integration by substitution.

SL Paper 1: Section B
No technology allowed

10. **a.** $\frac{\mathrm{d}y}{\mathrm{d}x}=3x^2\cos x-x^3\sin x$

b. $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2\sin x-(2x+1)\cos x}{\sin^2 x}$

c. $\frac{\mathrm{d}y}{\mathrm{d}x}=2xe^{(x^2+1)}$

d. $\frac{\mathrm{d}y}{\mathrm{d}x}=\ln(3x)+x\times\frac{1}{3x}\times 3=\ln(3x)+1$

11. **a.** $V = x^2 h$

$2x + h \le 1$ so, for a maximum $h = 1 - 2x$

$V = x^2 (1 - 2x) = x^2 - 2x^3$

$\frac{\mathrm{d}V}{\mathrm{d}x}=2x-6x^2=2x(1-3x)=0\Rightarrow x=0 \text{ or } \frac{1}{3}.$

Can reject $x = 0$ as $x > 0$ in the context of the problem.

Checking with a sign table (An alternative would be to look at the sign of the second derivative)

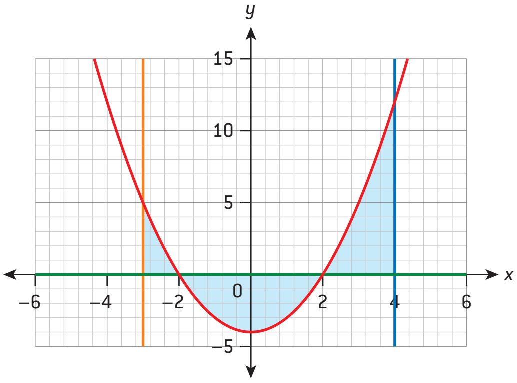
x	0	$0<x<\frac{1}{3}$	$\frac{1}{3}$	$x>\frac{1}{3}$
$\frac{dV}{dx}$	0	+ve	0	–ve

$x=\frac{1}{3}$ does give a maximum and the corresponding value of $h=\frac{1}{3}$

b. i. Maximum volume is $\left(\frac{1}{3}\right)^3=\frac{1}{27}\text{m}^3$

ii. surface area is $6\times\left(\frac{1}{3}\right)^2=\frac{2}{3}\text{m}^2$

12. a.



b. $\int x^2-4\,dx=\frac{x^3}{3}-4x$ $\quad\left[\frac{x^3}{3}-4x\right]_{-3}^{-2}=\left(\frac{-8}{3}+8\right)-(-9+12)=\frac{7}{3}$

$\left[\frac{x^3}{3}-4x\right]_{-2}^{2}=\left(\frac{8}{3}-8\right)-\left(\frac{-8}{3}+8\right)=\frac{-32}{3}$

$\left[\frac{x^3}{3}-4x\right]_{2}^{4}=\left(\frac{64}{3}-16\right)-\left(\frac{8}{3}-8\right)=\frac{32}{3}$

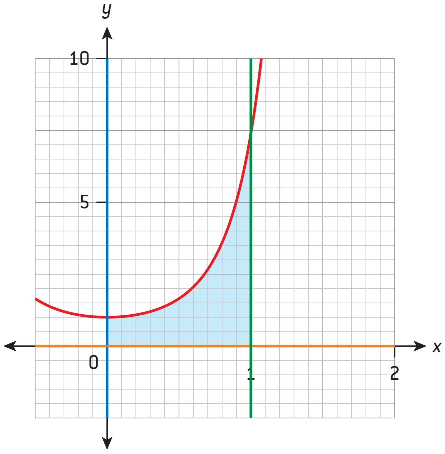
So total area is $\frac{7}{3}+\frac{32}{3}+\frac{32}{3}=\frac{71}{3}$

SL Paper 2: Section A Technology required

13. a. Require $\frac{dy}{dx}$ at $x=2$ which is -0.628 (3 sf)

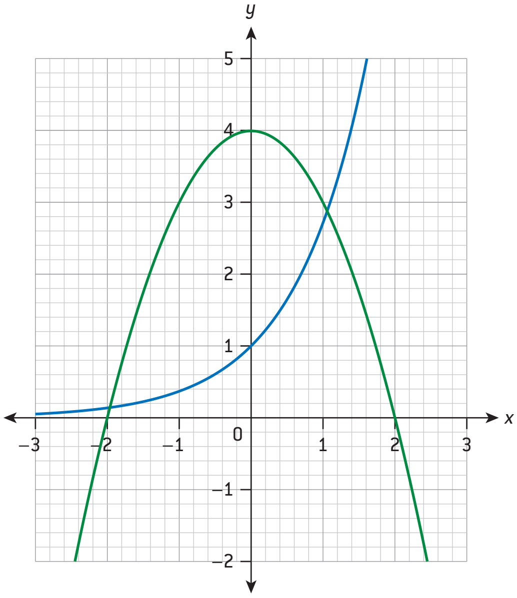
b. Gradient of normal is $\frac{-1}{-0.628279...}=1.59$ (3 sf)

14. a.



b. Definite integral required is $\int_0^1 e^{2x^2}\,dx = 2.36\,(3\,\text{sf})$

15. a.



b. Intersections (–1.96, 0.140) and (1.06, 2.88) (3 sf)

c. Require $\int_{-1.9646...}^{1.0580...} (4-x^2)-e^x\,dx = 6.43$ (3 sf)

16. a. $a(t)=\dfrac{dv}{dt}=12\cos 4t$

b. $s(t)=\int v\,dt=\dfrac{-3}{4}\cos 4t+c \qquad t=0, s=0 \Rightarrow c=\dfrac{3}{4} \qquad s(t)=\dfrac{-3}{4}\cos 4t+\dfrac{3}{4}$

c. $s(5)$ = 0.444 m (3 sf)

Require $\int_0^5 |3\sin 4t|\,dt = 9.44\,\text{m}$ (3 sf)

SL Paper 2: Section B
Technology required

17. a. $y(0) = 5 \Rightarrow c = 5$

$\frac{dy}{dx} = 3x^2 + 2ax + b = 0$ at a turning point so

$3 + 2a + b = 0,\ 48 + 8a + b = 0$ solving $a = \frac{-15}{2},\ b = 12$

b. Maximum is (1, 10.5), minimum is (4, –3)

c. $\frac{d^2y}{dx^2} = 6x - 15 = 0 \Rightarrow x = \frac{5}{2}$ point of inflexion is (2.5, 3.75)

The curve must have a point of inflexion between a maximum and a minimum.

HL Paper 1: Section A
No technology allowed

18. $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{((x+h)^2 - 4) - (x^2 - 4)}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$

$\lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} 2x + h = 2x$ (as expected)

19. Applying the substitution $u = 3x + 4$: $x = \frac{u-4}{3}$ $\frac{dx}{du} = \frac{1}{3}$

Limits: $x = -1 \Rightarrow u = 1,\ x = 0 \Rightarrow u = 4$

so integral becomes

$$\int_1^4 \frac{u-4}{3} \times \frac{1}{u^{\frac{1}{2}}} \times \frac{1}{3}\,du = \int_1^4 \frac{u^{\frac{1}{2}}}{9} - \frac{4u^{-\frac{1}{2}}}{9}\,du = \left[\frac{2u^{\frac{3}{2}}}{27} - \frac{8u^{\frac{1}{2}}}{9}\right]_1^4$$

$$= \left(\frac{16}{27} - \frac{16}{9}\right) - \left(\frac{2}{27} - \frac{8}{9}\right) = \frac{-10}{27}$$

20. a. $\lim_{x \to 0} \frac{\ln(1+x) - x}{x^2}$ is of the form $\frac{0}{0}$

It equals $\lim_{x \to 0} \frac{(1+x)^{-1} - 1}{2x}$, which is still of the form $\frac{0}{0}$ and so equals

$\lim_{x \to 0} \frac{-(1+x)^{-2}}{2} = \frac{-1}{2}$

b. $\lim_{x \to \infty} \frac{\ln x}{x^2}$ is of the form $\frac{\infty}{\infty}$. It equals $\lim_{x \to \infty} \frac{\frac{1}{x}}{2x} = \lim_{x \to \infty} \frac{1}{2x^2} = 0$

21. $\int e^y\,dy = \int \sec^2 x (\tan x)^3\,dx$ $e^y = \frac{\tan^4 x}{4} + c$

$y = 0$ when $x = 0$ gives $c = 1$ $e^y = \frac{\tan^4 x}{4} + 1$

$y = \ln\left(\frac{\tan^4 x}{4} + 1\right)$

HL Paper 1: Section B
No technology allowed

22. a. $\frac{dy}{dx}\tan x + y\sec^2 x + 2y\frac{dy}{dx}\arctan x + \frac{y^2}{1+x^2} = 0$

$\frac{dy}{dx}(\tan x + 2y\arctan x) = -y\sec^2 x - \frac{y^2}{1+x^2}$

$\frac{dy}{dx} = -\frac{\left(y\sec^2 x + \frac{y^2}{1+x^2}\right)}{(\tan x + 2y\arctan x)}$

b. $y = \operatorname{arccosec} x \Rightarrow \operatorname{cosec} y = x$

$-\operatorname{cosec} y\cot y\frac{dy}{dx} = 1 \Rightarrow \qquad \frac{dy}{dx} = \frac{-1}{\operatorname{cosec} y\cot y}$

Using $1+\cot^2 y = \operatorname{cosec}^2 y \Rightarrow \cot y = \sqrt{\operatorname{cosec}^2 y - 1}$

For $x > 1$, y is in the first quadrant so $\cot y$ will be positive

$\frac{dy}{dx} = \frac{-1}{x\sqrt{x^2-1}}$

23. a. $\int \frac{1}{1+x^2}dx = \arctan x + c$ from the formula book

b. $\int \frac{x}{1+x^2}dx = \frac{1}{2}\ln(1+x^2) + c$ by inspection

c. Using partial fractions:

$\frac{1}{1-x^2} = \frac{1}{(1+x)(1-x)} \equiv \frac{A}{1+x} + \frac{B}{1-x} \Rightarrow A(1-x) + B(1+x) \equiv 1$

Solving $A = \frac{1}{2}, B = \frac{1}{2}$ $\qquad \int \frac{\frac{1}{2}}{1+x} + \frac{\frac{1}{2}}{1-x}dx = \frac{1}{2}\ln|1+x| - \frac{1}{2}\ln|1-x| + c$

24. Using integration by parts and ILATE:

$\int 2x^2e^{2x}\,dx = x^2e^{2x} - \int 2xe^{2x}dx$

$u = 2x^2 \qquad v = \frac{e^{2x}}{2}$

$\frac{du}{dx} = 4x \qquad \frac{dv}{dx} = e^{2x}$

Using integration by parts again on $\int 2xe^{2x}dx$ and ILATE:

$u = 2x \qquad v = \frac{e^{2x}}{2}$

$\frac{du}{dx} = 2 \qquad \frac{dv}{dx} = e^{2x}$

$\int 2xe^{2x}dx = xe^{2x} - \int e^{2x}dx = xe^{2x} - \frac{e^{2x}}{2} + c$

So $\int 2x^2e^{2x}\,dx = x^2e^{2x} - xe^{2x} + \frac{e^{2x}}{2} + c$

25. $\frac{dy}{dx} = \frac{y}{x} + \sqrt{1 - \left(\frac{y}{x}\right)^2}$ $\qquad$ homogeneous, so let $v = \frac{y}{x}$

$v + x\frac{dv}{dx} = v + \sqrt{1-v^2}$ $\qquad \int \frac{1}{\sqrt{1-v^2}}dv = \int \frac{1}{x}dx$

$\arcsin v = \ln|x| + c$ $\qquad \arcsin\left(\frac{y}{x}\right) = \ln|x| + c$

$y = x\sin(\ln|x| + c)$

26. **a.** 1st order linear Integrating factor $= e^{\int 2x\,dx} = e^{x^2}$ $e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y = 4xe^{x^2}$

$e^{x^2}y = \int 4xe^{x^2}\,dx = 2e^{x^2} + c$ $y = 5$ when $x = 0 \Rightarrow c = 3$

$e^{x^2}y = 2e^{x^2} + 3$ $y = 2 + 3e^{-x^2}$

b. As $x \to \pm\infty,\ e^{-x^2} \to 0,\ y(x) \to 2$

27. **a.** $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$

b. $\int \arctan x\,dx = \int 1 \times \arctan x\,dx$ using integration by parts with ILATE

$u = \arctan x$ $v = x$

$\frac{du}{dx} = \frac{1}{1+x^2}$ $\frac{dv}{dx} = 1$

$\int \arctan x\,dx = x\arctan x - \int \frac{x}{1+x^2}\,dx = x\arctan x - \frac{1}{2}\ln(1+x^2) + c$

c. Integrating the Maclaurin expansion for $y = \arctan x$

$x\arctan x - \frac{1}{2}\ln(1+x^2) = c + \frac{x^2}{2} - \frac{x^4}{12} + \frac{x^6}{30} - \ldots$ putting $x = 0 \Rightarrow c = 0$

$x\arctan x - \frac{1}{2}\ln(1+x^2) = \frac{x^2}{2} - \frac{x^4}{12} + \frac{x^6}{30} - \ldots$

d. $x\arctan x = x^2 - \frac{x^4}{3} + \frac{x^6}{5} - \ldots$

e. Differentiating the answer to part **d.**

$\arctan x + \frac{x}{1+x^2} = 2x - \frac{4x^3}{3} + \frac{6x^5}{5} - \ldots$

f. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$

$\frac{1}{2}\ln(1+x^2) = \frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{6} - \ldots$

g. $x\arctan x - \frac{1}{2}\ln(1+x^2) = \left(x^2 - \frac{x^4}{3} + \frac{x^6}{5} - \ldots\right) - \left(\frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{6} - \ldots\right)$

$= \frac{x^2}{2} - \frac{x^4}{12} + \frac{x^6}{30} - \ldots,$ as required.

28. **a.** $f(x) = a^x$ $f'(x) = (\ln a)a^x$ $f''(x) = (\ln a)^2 a^x$ $f'''(x) = (\ln a)^3 a^x$

$f(0) = 1$ $f'(0) = (\ln a)$ $f''(0) = (\ln a)^2$ $f'''(0) = (\ln a)^3$

$f(x) = a^x = 1 + (\ln a)x + \frac{(\ln a)^2}{2!}x^2 + \frac{(\ln a)^3}{3!}x^3 + \ldots$

b. $a = e^{\ln a} \Rightarrow a^x = (e^{\ln a})^x = e^{(\ln a)x}$

$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots$ replacing x with $(\ln a)\,x$ gives

$a^x = 1 + (\ln a)x + \frac{(\ln a)^2}{2!}x^2 + \frac{(\ln a)^3}{3!}x^3 + \ldots,$ as required.

HL Paper 2: Section A
Technology required

29. **a.** Require $\int_0^1 2^{x^2}\,dx = 1.29$ (3sf)

b. Require $\int_0^1 \pi(2^{x^2})^2\,dx = \int_0^1 \pi 2^{2x^2}\,dx = 5.47$ (3sf)

30. Using $x_{n+1} = x_n + 0.25 \quad y_{n+1} = y_n + 0.25(\sqrt{x_n}\sqrt{y_n} + 1)$

$y(1) \simeq 2.68$ (3sf)

HL Paper 2: Section B
Technology required

31. Let the origin be where the wall meets the ground, the distance from the origin to the foot of the ladder be x and the distance from the origin to the top of the ladder be y.

a. $x^2 + y^2 = 13^2 \qquad 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \qquad \frac{dx}{dt} = 0.5 \qquad \frac{dy}{dt} = \frac{-0.5x}{y}$

When $x = 5$, $y = \sqrt{13^2 - 5^2} = 12 \qquad \frac{dy}{dt} = \frac{-0.5 \times 5}{12} = -0.208\,\text{m s}^{-1}$ (3 sf)

The negative indicates (as expected) that she is descending

b. When $y = 1$, $x = \sqrt{13^2 - 1^2} = \sqrt{168} \qquad \frac{dy}{dt} = \frac{-0.5 \times \sqrt{168}}{1} = -6.48\,\text{m s}^{-1}$ (3 sf)

c. As $x \to 13, y \to 0 \Rightarrow \frac{dx}{dt} \to -\infty$ which is impossible (not to mention very painful for Almu)

32. **a.** $y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \ldots$

$y(0) = 1 \qquad y' = x^2y^2 + 1 \qquad y'(0) = 1$

$y'' = 2xy^2 + x^2 2yy' \qquad y''(0) = 0$

$y''' = 2y^2 + 8xyy' + x^2(2(y')^2 + 2yy'') \qquad y'''(0) = 2$

$y(x) = 1 + x + \frac{x^2}{2!} \times 0 + \frac{x^3}{3!} \times 2 + \ldots \qquad y(x) = 1 + x + \frac{x^3}{3} + \ldots$

b. $y(1) \simeq 1 + 1 + \frac{1}{3} = \frac{7}{3}$

HL Paper 3: Technology required

33. a. i. $x = 2$ is the only solution to $y = 0$

So, only x-axis intercept is (2, 0)

ii. $\frac{dy}{dx} = 2(x-2)(x^2+3)+(x-2)^2 2x = (x-2)(4x^2-4x+6)$

$4x^2-4x+6=0$ has discriminant of $16-96=-80$ so this quadratic has no real roots and the only stationary point is (2, 0).

Looking at the sign table

x	$x<2$	2	$x>2$
$\frac{dy}{dx}$	–ve	0	+ve

So (2, 0) is a minimum.

b. i. $x = 2$ is the only solution to $y = 0$

So, only x-axis intercept is (2, 0)

ii. $\frac{dy}{dx} = 3(x-2)^2(x^2+3)+(x-2)^3 2x = (x-2)^2(5x^2-4x+9)$

$5x^2-4x+9=0$ has discriminant of $16-180=-164$ so this quadratic has no real roots and the only stationary point is (2, 0).

Looking at the sign table

x	$x<2$	2	$x>2$
$\frac{dy}{dx}$	+ve	0	+ve

So (2, 0) is a horizontal point of inflexion.

c. Let $p(x)=(x-c)^2 q(x)$ then $p'(x)=2(x-c)q(x)+(x-c)^2 q'(x)$

$p(c)=0, p'(c)=0+0=0$, so $(c, 0)$ is a stationary point.

d. Let $p(x)=(x-c)^3 q(x)$ then $p'(x)=3(x-c)^2 q(x)+(x-c)^3 q'(x)$

$p''(x)=6(x-c)q(x)+6(x-c)^2 q'(x)+(x-c)^3 q''(x)$

$p''(c)=0+0+0=0$

e. i. From part **c.** we know that $(c, 0)$ is a stationary point.

For $n = 2$

$p''(x)=2q(x)+4(x-c)q'(x)+(x-c)^2 q''(x)$

So $p''(c)=2q(c)\neq 0$ and so $(c, 0)$ cannot be a point of inflection.

$p'(x)=(x-c)(2q(x)+(x-c)q'(x))=(x-c)2q(x)+(x-c)^2 q'(x)$

Case 1: If $q(c)$ is positive then the sign table will be

x	$x<c$	$x=c$	$x>c$
$p'(x)$	–ve	0	+ve

so there is a minimum at $(c, 0)$

Case 2: If $q(c)$ is negative then the sign table will be:

x	$x < c$	$x = c$	$x > c$
$p'(x)$	+ve	0	−ve

so there is a maximum at $(c, 0)$

Note in both cases the $(x-c)2q(x)$ will be the dominant term as $x \to c$ so the sign of $q'(c)$ does not matter.

ii. From parts **c.** and **d.** we know that $p'(c) = 0$ and $p''(c) = 0$.

So, at the point $(c, 0)$ the graph is momentarily horizontal and straight.

$p'(x) = n(x-c)^{n-1}q(x) + (x-c)^n q'(x) = (x-c)^{n-1}(nq(x) + (x-c)q'(x))$

If n is odd and $q(c)$ is positive, then the sign table will be:

x	$x < c$	$x = c$	$x > c$
$p'(x)$	+ve	0	+ve

so there is a horizontal point of inflexion at $(c, 0)$.

If n is odd and $q(c)$ is negative, then the sign table will be:

x	$x < c$	$x = c$	$x > c$
$p'(x)$	−ve	0	−ve

so there is a horizontal point of inflexion at $(c, 0)$.

If n is even and $q(c)$ is positive, then the sign table will be:

x	$x < c$	$x = c$	$x > c$
$p'(x)$	−ve	0	+ve

so there is a flat bottom minimum at $(c, 0)$.

If n is even and $q(c)$ is negative, then the sign table will be:

x	$x < c$	$x = c$	$x > c$
$p'(x)$	+ve	0	−ve

so there is a flat top maximum at $(c, 0)$.

Note $n(x-c)^{n-1}q(x)$ will be the dominant term as $x \to c$.

Also note that in this question we are dealing with polynomials and so we know that all derivatives exist.

IB PREPARED MAA

WORKED SOLUTIONS AND MARKSCHEMES FOR THE PRACTICE EXAM PAPERS

Here are the worked solutions and markschemes for the practice exam papers from *IB Prepared MAA*.

These worked solutions are in a format that is similar to an IB markscheme. It shows how marks are to be gained in the different parts of a question. Examiners use annotations when marking and some of their abbreviations are shown here: M1 indicates a Method Mark, A1 an Achievement Mark and R1 a Reasoning Mark. If a mark is shown in brackets, e.g., (M1), it implies that the mark can be given even if the method has not been shown but is implied by subsequent student working. The abbreviation AG indicates "As Given" and is used at the end of a "show that …" problem. This means no marks would be given for this line and examiners know that they have to check the student's work carefully here, to see that the student has not just written down the required answer.

For direct access, click on the name of each practice exam paper.

SL PAPER 1 MARKSCHEME

Section A: Short answers

1. **a.** $u_1 + 4d = 18,\ u_1 + 9d = 38 \Rightarrow 5d = 20 \Rightarrow d = 4$ [(M1)A1]

[2 marks]

b. $d = 4 \Rightarrow u_1 + 16 = 18 \Rightarrow u_1 = 2$ [(M1)A1]

[2 marks]

c. $u_{101} = 2 + 100 \times 4 = 402$ [(M1)A1]

[2 marks]

[TOTAL 6 marks]

2. **a.** $\frac{dy}{dx} = \frac{1}{\cos^2 x}$ [M1A1]

at $x = \frac{\pi}{4}, \frac{dy}{dx} = 2$ [A1]

Equation of tangent is $y = 2x + c$

Through $\left(\frac{\pi}{4}, 1\right) \Rightarrow 1 = 2 \times \frac{\pi}{4} + c \Rightarrow c = 1 - \frac{\pi}{2}$ [M1]

Equation of tangent is

$y = 2x + 1 - \frac{\pi}{2}$ or $y - 1 = 2\left(x - \frac{\pi}{4}\right)$ [A1]

[5 marks]

b. Gradient of normal is $-\frac{1}{2}$ [A1]

Equation of normal is $y = -\frac{1}{2}x + d$.

Through $\left(\frac{\pi}{4}, 1\right) \Rightarrow 1 = -\frac{\pi}{8} + d \Rightarrow d = 1 + \frac{\pi}{8}$ [M1]

Equation of normal is

$y = -\frac{1}{2}x + 1 + \frac{\pi}{8}$ or $y - 1 = -\frac{1}{2}\left(x - \frac{\pi}{4}\right)$ [A1]

[3 marks]

[TOTAL 8 marks]

3. **a.** LHS $= \frac{1}{x+1} + \frac{x}{x+2} \equiv \frac{(x+2) + x(x+1)}{(x+1)(x+2)} \equiv \frac{x^2 + 2x + 2}{(x+1)(x+2)} =$ RHS [M1A1]

[2 marks]

b. $\frac{x^2 + 2x + 2}{(x+1)(x+2)} = \frac{10}{(x+1)(x+2)} \Rightarrow x^2 + 2x - 8 = 0$ [M1A1A1]

$\Rightarrow (x+4)(x-2) = 0 \Rightarrow x = -4$ or 2 [M1A1A1]

[6 marks]

[TOTAL 8 marks]

4. **a.** By inspection or substitution, $\int \cos x \sin^3 x \, dx = \frac{1}{4}(\sin^4 x) + c$ [(M1)A2]

[3 marks]

b. $\left[\frac{1}{4}(\sin^4 x)\right]_0^{\frac{\pi}{2}} = \frac{1}{4}$ [M1A1]

[2 marks]

[TOTAL 5 marks]

5. **a.** **i.** $\{x \in \mathbb{R}\}$ [A1]

ii. $\{y \in \mathbb{R} | y > 4\}$ [A1]

[2 marks]

b. Inverse given by $x = 3e^y + 4 \Rightarrow \frac{x-4}{3} = e^y \Rightarrow y = \ln\left(\frac{x-4}{3}\right)$ [M1A1]

$f^{-1}(x) = \ln\left(\frac{x-4}{3}\right)$ [A1]

[3 marks]

c. **i.** $\{x \in \mathbb{R} | x > 4\}$ [A1]

ii. $\{y \in \mathbb{R}\}$ [A1]

[2 marks]

[TOTAL 7 marks]

6.

D	1	2	3	4	5	6
1	0	1	2	3	4	5
2	1	0	1	2	3	4
3	2	1	0	1	2	3
4	3	2	1	0	1	2
5	4	3	2	1	0	1
6	5	4	3	2	1	0

[(M1)A1]

From the lattice diagram above (or otherwise) the probability distribution table for D is

d	0	1	2	3	4	5
$P(D = d)$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$

[M1A1]

$$E(D) = 0 \times \frac{6}{36} + 1 \times \frac{10}{36} + 2 \times \frac{8}{36} + 3 \times \frac{6}{36} + 4 \times \frac{4}{36} + 5 \times \frac{2}{36}$$

$$= \frac{10+16+18+16+10}{36} = \frac{70}{36} \left(= \frac{35}{18}\right)$$ [M1A1]

[TOTAL 6 marks]

Section B: Long answers

7. **a.** $f'(x) = 2x\cos x - x^2 \sin x$ [M1A1A1]

[3 marks]

b. $f''(x) = 2\cos x - 2x\sin x - 2x\sin x - x^2 \cos x = 2\cos x - 4x\sin x - x^2 \cos x$

[M1A1A1A1]

[4 marks]

c. **i.** 0

ii. 0

iii. 2 [A1A1A1]

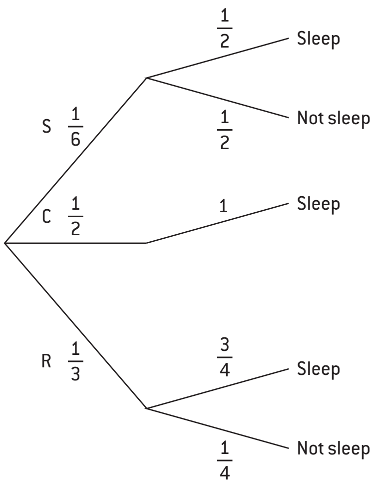
[3 marks]

d. Since $f'(0) = 0$ and $f''(0)$ is positive, the point (0, 0) must be a minimum. [R1A1A1]

[3 marks]

[TOTAL 13 marks]

8. **a.**



[M1A1A1]

[3 marks]

b. $\frac{1}{6}\times\frac{1}{2}+\frac{1}{2}\times 1+\frac{1}{3}\times\frac{3}{4}=\frac{1+6+3}{12}=\frac{10}{12}=\frac{5}{6}$ [M1A1]

[2 marks]

c. $P(C|\text{Sleep}) = \frac{P(C\cap \text{Sleep})}{P(\text{Sleep})} = \frac{\frac{1}{2}}{\frac{5}{6}} = \frac{3}{5}$ [M1A1]

[2 marks]

d. $P(S|\text{Not Sleep}) = \frac{P(S\cap \text{Not Sleep})}{P(\text{Not Sleep})} = \frac{\frac{1}{12}}{\frac{1}{6}} = \frac{1}{2}$ [M1A1]

[2 marks]

e. $P(S\cap \text{not } S) + P(C\cap \text{not } C) + P(R\cap \text{not } R)$

$=\frac{1}{6}\times\frac{5}{6}+\frac{1}{2}\times\frac{1}{2}+\frac{1}{3}\times\frac{2}{3}=\frac{5+9+8}{36}=\frac{22}{36}=\frac{11}{18}$ [M1A1A1]

[3 marks]

[TOTAL 12 marks]

9. a. i. $x=\frac{3}{2}$

ii. $y=\frac{1}{2}$ [A1A1]

[2 marks]

b. i. $(-1, 0)$

ii. $\left(0, \frac{-1}{3}\right)$ [A1A1]

[2 marks]

c. $f'(x)=\frac{1(2x-3)-(x+1)2}{(2x-3)^2}=\frac{-5}{(2x-3)^2}$ [M1A1]

[2 marks]

d. $f'(x)<0$ for all x, so graph is decreasing [R1AG]

[1 mark]

e. $f''(x)=\frac{20}{(2x-3)^3}$ [M1A1]

[2 marks]

f. If $x>\frac{3}{2}$, $f''(x)>0$, so graph is concave up [A1]

If $x<\frac{3}{2}$, $f''(x)<0$, so graph is concave down [A1]

[2 marks]

g. $\frac{1}{2}+\frac{\frac{5}{2}}{2x-3}\equiv\frac{\frac{1}{2}(2x-3)+\frac{5}{2}}{2x-3}\equiv\frac{x+1}{2x-3}$ [M1A1AG]

[2 marks]

h. $\int\frac{1}{2}+\frac{\frac{5}{2}}{2x-3}\mathrm{d}x=\frac{x}{2}+\frac{5}{4}\ln(2x-3)+c$ [M1A1]

[2 marks]

[TOTAL 15 marks]

SL PAPER 2 MARKSCHEME

Section A: Short answers

1. **a.** **i.** 7 [A1]

ii. 8 [A1]

[2 marks]

b. $Q_1 = 3, Q_3 = 8, IQR = 5$ [M1A1]

$8 + 1.5 \times 5 = 15.5$ so 16 is an outlier [R1A1]

[4 marks]

[TOTAL 6 marks]

2. $\dfrac{4}{\sin 40} = \dfrac{5}{\sin(A\hat{C}B \text{ or } C\hat{D}B)} \Rightarrow C\hat{D}B = 53.46...°, (A\hat{C}B = 126.5...°)$ [M1A1A1]

Triangle CBD isosceles $\Rightarrow CD = 2 \times 4 \cos 53.46...° = 4.76$ (3 s.f.) [R1M1A1A1]

[7 marks]

[TOTAL 7 marks]

3. General term is ${}^7C_r(x^2)^{7-r}(x^{-1})^r$ [M1A1]

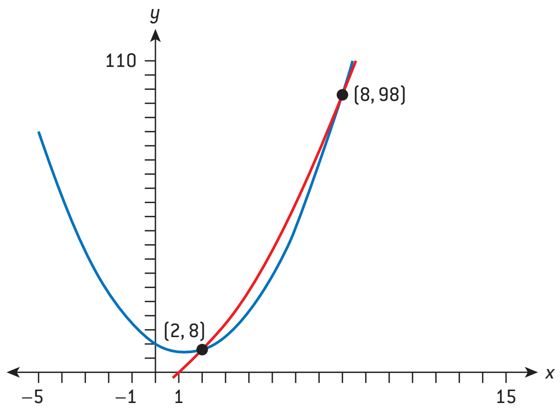
require $2(7 - r) - r = 2 \Rightarrow r = 4$ [A1A1]

Term is ${}^7C_4 x^2$, so coefficient is 35 [(A1)A1]

[6 marks]

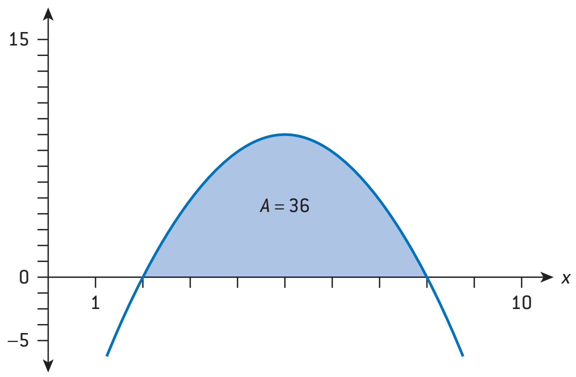
[TOTAL 6 marks]

4.



[(M1)A1]

$$A = \int_2^8 [(x^2 + 5x - 6) - (2x^2 - 5x + 10)]\,dx$$ [M1A1A1]



[(M1)]

$A = 36$ [A1]

OR (without GDC)

$2x^2 - 5x + 10 = x^2 + 5x - 6 \Rightarrow x^2 - 10x + 16 = 0$ [M1]

$(x-2)(x-8) = 0 \Rightarrow x = 2 \text{ or } 8$ [M1A1A1]

$A = \int_2^8 [(x^2 + 5x - 6) - (2x^2 - 5x + 10)]\mathrm{d}x = \int_2^8 [-x^2 + 10x - 16]\mathrm{d}x = 36$ [M1A1A1]

[7 marks]

[TOTAL 7 marks]

5. **a.** $500(1.04)^{10} = \$740.12$ [M1A1]

[2 marks]

b. Solving $(X(1.03)^5 - 200)(1.03)^5 - 300 = 0$ gives $X = 395.75$ [M1A1A2]

[4 marks]

[TOTAL 6 marks]

6. $\log_4 x - 3\log_x 4 - 2 = 0 \Rightarrow \log_4 x - \dfrac{3}{\log_4 x} - 2 = 0$ [M1A1]

$\Rightarrow (\log_4 x)^2 - 2\log_4 x - 3 = 0 \Rightarrow (\log_4 x - 3)(\log_4 x + 1) = 0$ [M1A1M1]

$\log_4 x = 3 \text{ or } -1 \Rightarrow x = 64 \text{ or } \dfrac{1}{4}$ [M1A1A1]

[8 marks]

[TOTAL 8 marks]

Section B: Long answers

7. **a.** $r = -0.856$ (3 sf) [A2]

[2 marks]

b. Quite strong negative linear correlation. [R1]

[1 mark]

c. $y = -0.733x + 25.9$ (3 sf) [M1A1A1]

[3 marks]

d. $-0.733 \times 31 + 25.9 = 3$ [M1A1]

[2 marks]

e. Using the line x on y, $-0.998 \times 11 + 30.6 = 20$ [M1M1A1]

[3 marks]

f. $(\bar{x}, \bar{y}) = (17.4, 13.1)$ (3 sf) [R1A1A1]

[3 marks]

[TOTAL 14 marks]

8. **a.** **i.** $P(24 < T < 27) = 0.533$ (3 sf) [(M1)A1]

ii. $P(26 < T) = 0.309$ (3 sf) [(M1)A1]

iii. $P(T < t) = 0.8 \Rightarrow t = 26.683\ldots$ [(M1)A1]

so cut off time is 26 minutes, 41 seconds [A1]

[7 marks]

b. **i.** $B(10, 0.8)$, $P(X = 7) = 0.201$ (3 sf) [(M1)A1]

ii. $P(X \geq 7) = 1 - P(X \leq 6) = 0.879$ (3 sf) [(M1)A1]

[4 marks]

c. $S \sim \mathrm{N}(\mu, 4^2)$, $P(S < 60) = 0.25$, $Z = \dfrac{S - \mu}{4} \sim \mathrm{N}(0, 1^2)$ [(M1)]

$P\left(Z < \dfrac{60 - \mu}{4}\right) = 0.25 \Rightarrow \dfrac{60 - \mu}{4} = -0.67448\ldots$ [M1A1]

$\mu = 62.697\ldots$ 62 minutes, 42 seconds [A1A1]

[5 marks]

[TOTAL 16 marks]

9. **a.** **i.** $1\ \mathrm{ms}^{-1}$ [A1]

ii. $-1\ \mathrm{ms}^{-1}$ [A1]

[2 marks]

b. **i.** $\int_0^{10} \sin\sqrt{t+1}\,\mathrm{d}t = 5.58$ m (3 sf) [M1A1]

ii. $\int_0^{10} \left|\sin\sqrt{t+1}\right|\mathrm{d}t = 5.78$ m (3 sf) [M1A2]

[5 marks]

c. Differentiating, acceleration is $\dfrac{1}{2\sqrt{t+1}}\cos\sqrt{t+1}$ [M1A2]

[3 marks]

[TOTAL 10 marks]

HL PAPER 1 MARKSCHEME

Section A: Short answers

1. a. $u_1 + 4d = 18, u_1 + 9d = 38 \Rightarrow 5d = 20 \Rightarrow d = 4$ [(M1)A1]

[2 marks]

b. $d = 4 \Rightarrow u_1 + 16 = 18 \Rightarrow u_1 = 2$ [(M1)A1]

[2 marks]

c. $u_{101} = 2 + 100 \times 4 = 402$ [(M1)A1]

[2 marks]

[TOTAL 6 marks]

2. a. $\frac{dy}{dx} = \frac{1}{\cos^2 x}$ [M1A1]

at $x = \frac{\pi}{4}, \frac{dy}{dx} = 2$ [A1]

Equation of tangent is $y = 2x + c$

Through $\left(\frac{\pi}{4}, 1\right) \Rightarrow 1 = 2 \times \frac{\pi}{4} + c \Rightarrow c = 1 - \frac{\pi}{2}$ [M1]

Equation of tangent is

$y = 2x + 1 - \frac{\pi}{2}$ or $y - 1 = 2\left(x - \frac{\pi}{4}\right)$ [A1]

[5 marks]

b. Gradient of normal is $-\frac{1}{2}$ [A1]

Equation of normal is $y = -\frac{1}{2}x + d$.

Through $\left(\frac{\pi}{4}, 1\right) \Rightarrow 1 = -\frac{\pi}{8} + d \Rightarrow d = 1 + \frac{\pi}{8}$ [M1]

Equation of normal is

$y = -\frac{1}{2}x + 1 + \frac{\pi}{8}$ or $y - 1 = -\frac{1}{2}\left(x - \frac{\pi}{4}\right)$ [A1]

[3 marks]

[TOTAL 8 marks]

3. a. By inspection or substitution, $\int \cos x \sin^3 x \, dx = \frac{1}{4}(\sin^4 x) + c$ [(M1)A2]

[3 marks]

b. $\left[\frac{1}{4}(\sin^4 x)\right]_0^{\frac{\pi}{2}} = \frac{1}{4}$ [M1A1]

[2 marks]

[TOTAL 5 marks]

4. a. i. $\{x \in \mathbb{R}\}$ [A1]

ii. $\{y \in \mathbb{R} | y > 4\}$ [A1]

[2 marks]

b. Inverse given by $x = 3e^y + 4 \Rightarrow \frac{x-4}{3} = e^y \Rightarrow y = \ln\left(\frac{x-4}{3}\right)$ [M1A1]

$f^{-1}(x) = \ln\left(\frac{x-4}{3}\right)$ [A1]

[3 marks]

c. i. $\{x \in \mathbb{R} | x > 4\}$ [A1]

ii. $\{y \in \mathbb{R}\}$ [A1]

[2 marks]

[TOTAL 7 marks]

5. [(M1)A1]

D	1	2	3	4	5	6
1	0	1	2	3	4	5
2	1	0	1	2	3	4
3	2	1	0	1	2	3
4	3	2	1	0	1	2
5	4	3	2	1	0	1
6	5	4	3	2	1	0

From the lattice diagram above (or otherwise) the probability distribution table for D is

d	0	1	2	3	4	5
$P(D = d)$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$

[M1A1]

$$E(D) = 0 \times \frac{6}{36} + 1 \times \frac{10}{36} + 2 \times \frac{8}{36} + 3 \times \frac{6}{36} + 4 \times \frac{4}{36} + 5 \times \frac{2}{36} = \frac{10+16+18+16+10}{36}$$

$$= \frac{70}{36}\left(= \frac{35}{18}\right)$$ [M1A1]

[TOTAL 6 marks]

6. $w^* = 1 + 4i \Rightarrow w = 1 - 4i$ [(A1)]

$$\frac{z}{w} = \frac{2+3i}{1-4i} \times \frac{1+4i}{1+4i} = \frac{2+8i+3i-12}{17} = \frac{-10}{17} + \frac{11}{17}i$$ [M1A1A1A1]

[5 marks]

[TOTAL 5 marks]

7. $$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{((x+h)^2 + (x+h) + 2) - (x^2 + x + 2)}{h}$$ [M1A1]

$$= \lim_{h \to 0} \frac{2xh + h^2 + h}{h} = \lim_{h \to 0} \frac{h(2x+h+1)}{h} = \lim_{h \to 0}(2x + 1 + h) = 2x + 1$$ [A1(A1)A1]

[5 marks]

[TOTAL 5 marks]

8. $p(1) = 1 + 1 + a + b = 7 \Rightarrow a + b = 5$ [M1A1]

$p(-2) = -8 + 4 - 2a + b = -8 \Rightarrow -2a + b = -4$ [A1]

Solving simultaneously $\Rightarrow a = 3, b = 2$ [M1A1A1]

[6 marks]

[TOTAL 6 marks]

9. a. $\mathbf{a} \bullet \mathbf{b} = 0 \Rightarrow 2 + 2 - k = 0 \Rightarrow k = 4$ [M1A1A1]

[3 marks]

b. $\mathbf{c} = \lambda \mathbf{b} \Rightarrow \lambda = 2 \Rightarrow l = -2$ [M1A1]

[2 marks]

c. $\mathbf{b} \times \mathbf{c} = \mathbf{0}$ because $\mathbf{c}$ and $\mathbf{b}$ are parallel [R1]

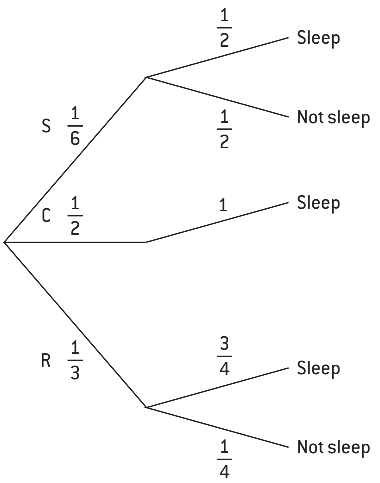
$\Rightarrow \mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = 0$ [A1]

[2 marks]

[TOTAL 6 marks]

Section B: Long answers

10. a.



[M1A1A1]

[3 marks]

b. $\frac{1}{6} \times \frac{1}{2} + \frac{1}{2} \times 1 + \frac{1}{3} \times \frac{3}{4} = \frac{1+6+3}{12} = \frac{10}{12} = \frac{5}{6}$ [M1A1]

[2 marks]

c. $P(C|\text{Sleep}) = \frac{P(C \cap \text{Sleep})}{P(\text{Sleep})} = \frac{\frac{1}{2}}{\frac{5}{6}} = \frac{3}{5}$ [M1A1]

[2 marks]

d. $P(S|\text{Not Sleep}) = \frac{P(S \cap \text{Not Sleep})}{P(\text{Not Sleep})} = \frac{\frac{1}{12}}{\frac{1}{6}} = \frac{1}{2}$ [M1A1]

[2 marks]

e. $P(S \cap \text{not } S) + P(C \cap \text{not } C) + P(R \cap \text{not } R)$

$= \frac{1}{6} \times \frac{5}{6} + \frac{1}{2} \times \frac{1}{2} + \frac{1}{3} \times \frac{2}{3} = \frac{5+9+8}{36} = \frac{22}{36} = \frac{11}{18}$ [M1A1A1]

[3 marks]

[TOTAL 12 marks]

11. a. $x^2+x-6=(x+3)(x-2)$ [A1]

$\dfrac{3x+4}{x^2+x-6}\equiv\dfrac{A}{x+3}+\dfrac{B}{x-2}\Rightarrow 3x+4\equiv A(x-2)+B(x+3)$ [M1A1]

$x=-3\Rightarrow A=1,\ x=2\Rightarrow B=2$ [(M1)A1]

$f(x)=\dfrac{1}{x+3}+\dfrac{2}{x-2}$ [A1]

$\displaystyle\int\left(\frac{1}{x+3}+\frac{2}{x-2}\right)dx=\ln(x+3)+2\ln(x-2)+c$ [M1A1]

[8 marks]

b. $f'(x)=-\dfrac{1}{(x+3)^2}-\dfrac{2}{(x-2)^2}$ [M1A1]

As squares are always non negative, this expression is always negative and hence the function is always decreasing. [R1]

[3 marks]

[TOTAL 11 marks]

12. a. Let $P(n)$ be the statement that $\displaystyle\sum_{i=1}^{n} i^3=\frac{1}{4}n^2(n+1)^2$.

For $n=1$, LHS $=1^3=1$, RHS $=\dfrac{1^2\times 2^2}{4}=1$, so $P(1)$ is true. [M1A1]

Assume $P(k)$ is true and attempt to prove for $P(k+1)$ [M1]

LHS of $P(k+1)=\displaystyle\sum_{i=1}^{k+1} i^3=\sum_{i=1}^{k} i^3+(k+1)^3=\frac{1}{4}k^2(k+1)^2+(k+1)^3$ [M1A1A1]

$=\dfrac{1}{4}(k+1)^2(k^2+4k+4)=\dfrac{1}{4}(k+1)^2(k+2)^2=$ RHS of $P(k+1)$ [M1A1]

So since $P(1)$ is true and $P(k)$ true implies $P(k+1)$ true, by the principle of mathematical induction the statement has been proved for all $n\in\mathbb{Z}^+$. [R1]

[9 marks]

b. $\displaystyle\sum_{i=1}^{n} i$ is an arithmetic sequence with $u_1=d=1$ so $S_n=\dfrac{n}{2}(1+n)$ [M1A1]

Hence $\displaystyle\left(\sum_{i=1}^{n} i\right)^2=\frac{n^2(n+1)^2}{4}$ as required. **[2 marks]**

c. Since "odds" equals "all" minus "evens"

$\displaystyle\sum_{i=1}^{n}(2i-1)^3=\sum_{i=1}^{2n} i^3-\sum_{i=1}^{n}(2i)^3=\sum_{i=1}^{2n} i^3-8\sum_{i=1}^{n} i^3$ [M1A1]

$\displaystyle=\frac{(2n)^2(2n+1)^2}{4}-8\left(\frac{n^2(n+1)^2}{4}\right)$ [A1]

[Note: the step above uses $\displaystyle\sum_{i=1}^{2n} i^3=\frac{(2n)^2(2n+1)^2}{4}$ and $\displaystyle\sum_{i=1}^{n} i^3=\frac{1}{4}n^2(n+1)^2$ by the formula proved in part **a.**]

Simplifying gives

$\displaystyle\sum_{i=1}^{n}(2i-1)^3=n^2(2n+1)^2-2n^2(n+1)^2=n^2(2n^2-1)$ [A1A1]

[5 marks]

[TOTAL 16 marks]

13. a. $\ln y = \ln x^x = x\ln x$ [A1]

$\frac{1}{y}\frac{dy}{dx} = \ln x + 1$ [M1A1A1]

$\frac{dy}{dx} = y(\ln x + 1) = x^x(\ln x + 1)$ [A1]

[5 marks]

b. i. $x^x(\ln x + 1) = 0 \Rightarrow \ln x + 1 = 0 \Rightarrow \ln x = -1 \Rightarrow x = e^{-1}$ [M1A1]

So point is $(e^{-1}, e^{-e^{-1}})$ [A1]

ii.

x	$x < e^{-1}$	$x = e^{-1}$	$x > e^{-1}$
$\frac{dy}{dx}$	–ve	0	+ve

Sign diagram shows that the point is a minimum. [M1A1]

[5 marks]

c. $\lim_{x\to 0}(x\ln x) = \lim_{x\to 0}\frac{\ln x}{\frac{1}{x}}$ which is of the form $\frac{-\infty}{\infty}$ [M1]

$\lim_{x\to 0}\frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0}(-x) = 0$ M1A1A1

[4 marks]

d. $\ln x^x \to 0 \Rightarrow x^x \to 1$ as $x \to 0$ [M1A1]

[2 marks]

[TOTAL 16 marks]

HL PAPER 2 MARKSCHEME

Section A: Short answers

1. $\frac{4}{\sin 40} = \frac{5}{\sin(A\hat{C}B \text{ or } C\hat{D}B)} \Rightarrow C\hat{D}B = 53.46...°, (A\hat{C}B = 126.5...°)$ [M1A1A1]

Triangle CBD isosceles $\Rightarrow CD = 2 \times 4\cos 53.46...° = 4.76$ (3 s.f.) [R1M1A1A1]

[7 marks]

[TOTAL 7 marks]

2. General term is ${}^7C_r (x^2)^{7-r}(x^{-1})^r$ [M1A1]

require $2(7-r) - r = 2 \Rightarrow r = 4$ [A1A1]

Term is ${}^7C_4 x^2$, so coefficient is 35 [(A1)A1]

[6 marks]

[TOTAL 6 marks]

3. **a.** $500(1.04)^{10} = \$740.12$ [M1A1]

[2 marks]

b. Solving $(X(1.03)^5 - 200)(1.03)^5 - 300 = 0$ gives $X = 395.75$ [M1A1A2]

[4 marks]

[TOTAL 6 marks]

4. $\log_4 x - 3\log_x 4 - 2 = 0 \Rightarrow \log_4 x - \frac{3}{\log_4 x} - 2 = 0$ [M1A1]

$\Rightarrow (\log_4 x)^2 - 2\log_4 x - 3 = 0 \Rightarrow (\log_4 x - 3)(\log_4 x + 1) = 0$ [M1A1M1]

$\log_4 x = 3 \text{ or } -1 \Rightarrow x = 64 \text{ or } x = \frac{1}{4}$ [M1A1A1]

[8 marks]

[TOTAL 8 marks]

5. **a.** **i.** $\pi\int_0^{\pi^2} (\sin\sqrt{x})^2 \, dx$ [M1A1]

ii. 15.5 (3 sf) [A2]

[4 marks]

b. **i.** Maximum of sin is 1, so maximum radius is 1 [R1]

ii. Occurs when $\sqrt{x} = \frac{\pi}{2} \Rightarrow x = \frac{\pi^2}{4}$ [M1A1]

[3 marks]

[TOTAL 7 marks]

6. a. $A = \pi r^2 - l^2$ [M1A1]

$\frac{dA}{dt} = 2\pi r\frac{dr}{dt} - 2l\frac{dl}{dt} = 0.4\pi r - 0.2l$ [M1A1A1]

when $r = 2$ and $l = 1$,

$\frac{dA}{dt} = 0.8\pi - 0.2 = 2.31\,\text{m}^2\ \text{s}^{-1}$ (3 sf) [M1A1]

[7 marks]

b. $10 = 0.4\pi r - 0.2l \Rightarrow l = 2\pi r - 50$ [M1A1]

[2 marks]

[TOTAL 9 marks]

7. $f'(0) = 1\times 2 + 3 = 5$ [A1]

$f''(x) = f(x) + (x+1)f'(x)$ [M1A1]

$f''(0) = 2 + 5 = 7$ [A1]

$f'''(x) = 2f'(x) + (x+1)f''(x)$ [M1A1]

$f'''(0) = 2\times 5 + 7 = 17$ [A1]

$f(x) = 2 + 5x + \frac{7}{2}x^2 + \frac{17}{6}x^3 + \ldots$ [A1]

[8 marks]

[TOTAL 8 marks]

Section B: Long answers

8. a. $r = -0.856$ (3 sf) [A2]

[2 marks]

b. Quite strong negative linear correlation. [R1]

[1 mark]

c. $y = -0.733x + 25.9$ (3 sf) [M1A1A1]

[3 marks]

d. $-0.733\times 31 + 25.9 = 3$ [M1A1]

[2 marks]

e. Using the line x on y, $-0.998\times 11 + 30.5 = 20$ [M1M1A1]

[3 marks]

f. $(\bar{x}, \bar{y}) = (17.4, 13.1)$ (3 sf) [R1A1A1]

[3 marks]

[TOTAL 14 marks]

9. $\frac{dy}{dx}=\frac{1+\frac{y}{x}}{1-\frac{y}{x}}$ so it is homogeneous. [A1R1]

Letting $v=\frac{y}{x}$

$y=xv$ [M1]

$x\frac{dv}{dx}+v=\frac{1+v}{1-v}$ [A1]

$x\frac{dv}{dx}=\frac{1+v^2}{1-v}$ [A1]

$\int\frac{1-v}{1+v^2}dv=\int\frac{1}{x}dx$ [M1]

$\int\left(\frac{1}{1+v^2}-\frac{v}{1+v^2}\right)dv=\ln x+c$ [M1A1]

$\arctan v-\frac{1}{2}\ln(1+v^2)=\ln x+c$ [A1A1]

$\arctan\left(\frac{y}{x}\right)=\frac{1}{2}\ln\left(1+\left(\frac{y}{x}\right)^2\right)+\ln x+c$ [M1]

$\arctan\left(\frac{y}{x}\right)=\frac{1}{2}\ln(x^2+y^2)+c$ [A1]

$x=1, y=1\Rightarrow c=\frac{\pi}{4}-\frac{1}{2}\ln 2$ [M1A1]

$\arctan\left(\frac{y}{x}\right)=\frac{1}{2}\ln(x^2+y^2)+\frac{\pi}{4}-\frac{1}{2}\ln 2$ [A1]

[15 marks]

[TOTAL 15 marks]

10. a. $\int_0^1 ae^x\,dx=1\Rightarrow\left[ae^x\right]_0^1=a(e-1)=1\Rightarrow a=\frac{1}{e-1}$ [M1A1AG]

[2 marks]

b. i. $\mu=\int_0^1\frac{xe^x}{e-1}dx=0.582\text{ (3 sf)}\left(=\frac{1}{e-1}\right)$ [(M1)A2]

ii. $\sigma^2=\int_0^1\frac{x^2e^x}{e-1}dx-(0.582\ldots)^2=0.0793\text{ (3 sf)}$ [M1A2]

[6 marks]

c. $\int_0^M\frac{e^x}{e-1}dx=\frac{1}{2}\Rightarrow\left[\frac{e^x}{e-1}\right]_0^M=\frac{e^M-1}{e-1}=\frac{1}{2}$ [M1A1]

$2e^M-2=e-1\Rightarrow e^M=\frac{e+1}{2}\Rightarrow M=\ln\left(\frac{e+1}{2}\right)\text{ (= 0.620 (3 sf))}$ [M1A1]

[4 marks]

[TOTAL 12 marks]

11. **a.** Plane is perpendicular to $\begin{pmatrix}2\\2\\1\end{pmatrix}$, line is parallel to $\begin{pmatrix}3\\2\\1\end{pmatrix}$ [R1R1]

Using the dot product to find the angle between these two vectors [M1]

$6+4+1=\sqrt{4+4+1}\sqrt{9+4+1}\cos\theta$ [A1]

$\cos\theta=\dfrac{11}{3\sqrt{14}}\Rightarrow\theta=11.490...$ [(A1)A1]

So angle between line and plane is 78.5° (3 sf) [A1]

[7 marks]

b. Since line and plane are not parallel they must cross. [R1]

Points on the line have the form $x=1+3t,\ y=2+2t,\ z=3+t$ [A1]

Substituting into the plane:
$2(1+3t)+2(2+2t)+(3+t)=9\Rightarrow 11t=0\Rightarrow t=0$ [M1A1]

Intersection point is (1, 2, 3) [A1]

[5 marks]

c. Let F be the foot of the perpendicular from P to the plane.

$\overrightarrow{PF}=\lambda\begin{pmatrix}2\\2\\1\end{pmatrix},\quad \overrightarrow{OF}=\begin{pmatrix}2\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}2\\2\\1\end{pmatrix}$ [R1A1]

F lies in the plane so $2(2+2\lambda)+2(1+2\lambda)+(1+\lambda)=9\Rightarrow\lambda=\dfrac{2}{9}$ [M1A1]

Distance is $\left|\overrightarrow{PF}\right|=\dfrac{2}{9}\sqrt{4+4+1}=\dfrac{2}{3}$ [M1A1]

(Note there are several other methods.) **[6 marks]**

[TOTAL 18 marks]

HL PAPER 3 MARKSCHEME

Section A: Short answers

1. **a.** **i.** $f'(x) = e^x \sin x + e^x \cos x$ [M1A1]

ii. $f''(x) = e^x \sin x + e^x \cos x + e^x \cos x - e^x \sin x = 2e^x \cos x$ [M1A1]

iii. $f'''(x) = 2e^x \cos x - 2e^x \sin x = 2e^x(\cos x - \sin x)$ [M1A1]

iv. $f^{(4)}(x) = 2e^x \cos x - 2e^x \sin x - 2e^x \sin x - 2e^x \cos x = -4e^x \sin x$ [M1A1]

v. $f^{(4n)}(x) = (-4)^n e^x \sin x$ [A2]

vi. $f^{(4n+1)}(x) = (-4)^n (e^x \sin x + e^x \cos x)$ [A1]

vii. $f^{(4n+2)}(x) = (-4)^n (2e^x \cos x)$ [A1]

viii. $f^{(4n+3)}(x) = (-4)^n (2e^x \cos x - 2e^x \sin x)$ [A1]

[13 marks]

b. **i.** $f'(x) = g'(x) \times h(x) + g(x) \times h'(x)$ [A1]

ii. $f''(x) = g''(x) \times h(x) + 2g'(x) \times h'(x) + g(x) \times h''(x)$ [M1A1]

iii. $f'''(x) = g'''(x) \times h(x) + 3g''(x) \times h'(x) + 3g'(x) \times h''(x) + g(x) \times h'''(x)$

[M1A1]

[5 marks]

c. $f(x) = e^{\lambda x} e^x = e^{(1+\lambda)x}$ [A1]

$f^{(n)}(x) = (1+\lambda)^n e^{(1+\lambda)x}$ [M1A1]

$g^{(n-i)}(x) = \lambda^{n-i} e^{\lambda x}$, $h^{(i)}(x) = e^x$ [A1A1]

So equation becomes

$(1+\lambda)^n e^{(1+\lambda)x} = \sum_{i=0}^{n} a_i \lambda^{n-i} e^{\lambda x} \times e^x = \sum_{i=0}^{n} a_i \lambda^{n-i} e^{(1+\lambda)x}$ [M1A1]

So $(1+\lambda)^n = \sum_{i=0}^{n} a_i \lambda^{n-i}$ [A1]

Giving $a_i = {}^nC_i$ from the binomial expansion of $(1+\lambda)^n$ [A1R1]

[10 marks]

[TOTAL 28 marks]

2. **a.** $|z| = 1$ [A1]

[1 mark]

b. **i.** $z^n + \frac{1}{z^n} = (\text{cis}\,\theta)^n + (\text{cis}\,\theta)^{-n} = \text{cis}(n\theta) + \text{cis}(-n\theta)$ [M1A1]

$= \cos(n\theta) + i\sin(n\theta) + \cos(-n\theta) + i\sin(-n\theta)$

$= \cos(n\theta) + i\sin(n\theta) + \cos(n\theta) - i\sin(n\theta)$

$= 2\cos n\theta$ [M1AG]

ii. $(\text{cis}\theta)^n - (\text{cis}\theta)^{-n} = \text{cis}(n\theta) - \text{cis}(-n\theta)$ [A1]

$= \cos(n\theta) + \mathrm{i}\sin(n\theta) - \cos(-n\theta) - \mathrm{i}\sin(-n\theta)$

$= \cos(n\theta) + \mathrm{i}\sin(n\theta) - \cos(n\theta) + \mathrm{i}\sin(n\theta)$

$= 2\mathrm{i}\sin n\theta$ [M1AG]

[5 marks]

c. i. $\left(z+\frac{1}{z}\right)^2 = z^2 + \frac{1}{z^2} + 2 = 2\cos 2\theta + 2 = 2(1+\cos 2\theta)$ [A1]

Hence, $(2\cos\theta)^2 = 4\cos^2\theta = 2(\cos 2\theta + 1)$ [M1A1]

$\Rightarrow \cos^2\theta = \frac{1+\cos 2\theta}{2}$ [AG]

ii. $\left(z-\frac{1}{z}\right)^2 = z^2 - 2 + \frac{1}{z^2} = 2(\cos 2\theta - 1)$ [A1]

$(2\mathrm{i}\sin\theta)^2 = -4\sin^2\theta = 2(\cos 2\theta - 1)$ [A1]

$\Rightarrow \sin^2\theta = \frac{1-\cos 2\theta}{2}$ [AG]

[5 marks]

d. i. $\left(z+\frac{1}{z}\right)^3 = z^3 + 3z + \frac{3}{z} + \frac{1}{z^3}$

$= z^3 + \frac{1}{z^3} + 3\left(z+\frac{1}{z}\right)$ [A1]

$(2\cos\theta)^3 = 2\cos 3\theta + 3\times 2\cos\theta \Rightarrow 8\cos^3\theta = 2\cos 3\theta + 6\cos\theta$ [M1A1]

$\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$ [A1]

ii. $\left(z+\frac{1}{z}\right)^4 = z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4}$

$= z^4 + \frac{1}{z^4} + 4\left(z^2+\frac{1}{z^2}\right) + 6$ [A1]

$(2\cos\theta)^4 = 2\cos 4\theta + 4\times 2\cos 2\theta + 6 \Rightarrow 16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$

[A1]

$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$ [A1]

[7 marks]

e. i. $\left(z-\frac{1}{z}\right)^3 = z^3 - 3z + \frac{3}{z} - \frac{1}{z^3}$

$= z^3 - \frac{1}{z^3} - 3\left(z-\frac{1}{z}\right)$ [A1]

$(2\mathrm{i}\sin\theta)^3 = 2\mathrm{i}\sin 3\theta - 3\times 2\mathrm{i}\sin\theta \Rightarrow -8\mathrm{i}\sin^3\theta = 2\mathrm{i}\sin 3\theta - 6\mathrm{i}\sin\theta$ [A1]

$\sin^3\theta = \frac{-1}{4}\sin 3\theta + \frac{3}{4}\sin\theta$ [A1]

ii. $\left(z-\frac{1}{z}\right)^4 = z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4}$

$= z^4 + \frac{1}{z^4} - 4\left(z^2+\frac{1}{z^2}\right) + 6$ [A1]

$(2\mathrm{i}\sin\theta)^4 = 2\cos 4\theta - 4\times 2\cos 2\theta + 6 \Rightarrow 16\sin^4\theta = 2\cos 4\theta - 8\cos 2\theta + 6$

[A1]

$\sin^4\theta = \frac{1}{8}\cos 4\theta - \frac{1}{2}\cos 2\theta + \frac{3}{8}$ [A1]

[6 marks]

f. $\cos^4\theta - \sin^4\theta = (\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta)$ [M1A1]

$= \cos^2\theta - \sin^2\theta = \cos 2\theta$ [AG]

From parts **d. ii.** and **e. ii.**

$\cos^4\theta - \sin^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8} - \left(\frac{1}{8}\cos 4\theta - \frac{1}{2}\cos 2\theta + \frac{3}{8}\right)$ [M1]

$= \cos 2\theta$, confirming the result [AG]

[3 marks]

[TOTAL 27 marks]